Advanced Microeconomics (ES30025)

Mathematics Review 3: Monotonic Transformations

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I. Introduction

A utility function is simply a way of assigning a preference ordering of consumption bundles. It does this by assigning numbers to consumption bundles so that the bundles that are more preferred get a higher number than those bundles that are less preferred. Mathematically, the utility function is a function that maps from the consumption set to set of the real numbers. Hence, a utility function assigns a unique number to each bundle in such a way that bundles that are more preferred by consumers’ are assigned a higher number. Note that it does not matter how much higher one number is from another – we measure utility ordinally rather than cardinally.

More formally, a utility function represents a preference ordering if for any consumption bundles \( x \) and \( y \), \( x \succ y \iff u(x) > u(y) \). Thus, a utility function represents a preference relation if and only if (i.e. iff) the assignment of numbers retains the ranking order (according to the preference relation) of the consumption bundles.

A utility function representing a preference ordering is not unique. The only requirement for a particular assignment of numbers to be classified as ‘a utility function representing a preference ordering’ is that it must retain the rank order of consumption bundles in terms of the preference of the consumer. Thus, any other assignment of numbers that keeps the ranking of bundles intact will also be a utility representation. For example, both the columns in Table 1 represent the same preference relation over \((x, y, z)\).

<table>
<thead>
<tr>
<th>Consumption Bundle</th>
<th>( u(\ ) )</th>
<th>( v(\ ) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>( y )</td>
<td>1</td>
<td>-597</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>-356678</td>
</tr>
</tbody>
</table>

Table 1: Preference Orderings

It is important to note that once we have a particular utility function representing a preference ordering, we can derive an infinite number of other utility functions representing the same preference ordering. For example, multiplying the original utility function by any positive real number will give us a new utility function that preserves the ranking order of bundles and hence is a utility representation.

II. Monotonic Transformations

A (positive) monotonic transformation is way of transforming one set of numbers into another set of numbers so that the rank order of the original set of numbers is preserved. It is thus a function, \( f \), mapping real numbers into real numbers, which satisfies the property that if \( x > y \), then \( f(x) > f(y) \). Simply, it is a strictly increasing function. Thus, any (positive) monotonic transformation of the utility function is also a utility function representing the same preference relation because such a transformation preserves the rank order of the
original utility numbers and hence the ranking order of the bundles based on the preference relation. So, if we begin with the utility function \( u(x) \) and then use the (positive) monotonic transformation \( f \) to get a new function \( v(x) = f \cdot u(x) \), then \( v(x) \) is also a utility function representing the same preference relation as \( u(x) \).

Monotonic transformations are useful as they can often help us in solving otherwise cumbersome optimizations – we can make a utility function easier to work without affecting the optimal choice. Consider the following:

\[
\max_x u(x) = 1 + 10x - x^2
\]

\[
\Rightarrow \quad \frac{du(x)}{dx} = 10 - 2x^* = 0 \quad \text{ (1)}
\]

\[
\Rightarrow \quad x^* = 5
\]

The solution to this problem is shown in Figure 1:

![Figure 1: Optimizing \( u(x) \)](image)

Now consider optimizing a monotonic transformation of \( u(x) \) vis:
\[
\max_x v(x) = 2 \left[ u(x) \right] = 2 + 20x - 2x^2 \\
\Rightarrow \\
\frac{dv(x)}{dx} = 20 - 4x^* = 0 \\
\Rightarrow \\
x^* = 5
\]

The solution to this optimization is shown in Figure 2:

![Figure 2: Optimizing \( v(x) = 2[u(x)] \)](image)

It is apparent that the optimal choice of \( x \) does not change – all that is affected are the units in which we measure utility.

A monotonic transformation whereby we multiply the original utility function by 2 is not going to help us too much in terms of simplifying optimization problems. But note the following:

\[
\max_x w(x) = \ln \left[ u(x) \right] = \ln \left( 1 + 10x - x^2 \right) \\
\Rightarrow \\
\frac{dw(x)}{dx} = \frac{10 - 2x^*}{1 + 10x - (x^*)^2} = 0 \\
\Rightarrow \\
x^* = 5
\]
Taking logarithms is a particularly helpful monotonic transformation as it can turn a non-linear problem into a linear problem. For example, assume the consumer’s utility function is Cobb-Douglas of the form:

$$u(x_1, x_2) = x_1^c x_2^d$$  \hspace{1cm} (4)

Then we can take a logarithm monotonic transformation to yield:

$$w(x_1, x_2) = \ln[u(x_1, x_2)] = \ln(x_1^c x_2^d) = c \ln x_1 + d \ln x_2$$  \hspace{1cm} (5)

Equation (5) is thus linear and so is potentially easier to deal with than equation (4). Consider also the following monotonic transformation of (4):

$$\omega(x_1, x_2) = \left[ u(x_1, x_2) \right]^{\frac{1}{c+d}} = \left( x_1^c x_2^d \right)^{\frac{1}{c+d}} = x_1^{\frac{c}{c+d}} x_2^{\frac{d}{c+d}}$$  \hspace{1cm} (6)

Now define a new number $a = \left[ \frac{c}{c+d} \right]$ such that:

$$\omega(x_1, x_2) = x_1^{\frac{c}{c+d}} x_2^{\frac{d}{c+d}} = x_1^{a} x_2^{1-a}$$  \hspace{1cm} (7)

Thus, we can always take a monotonic transformation of the Cobb-Douglas utility function that makes the exponents sum to one.