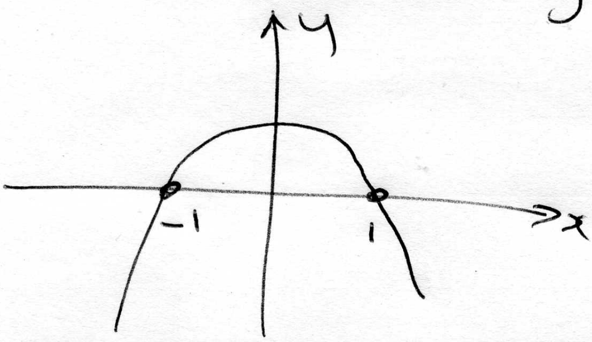
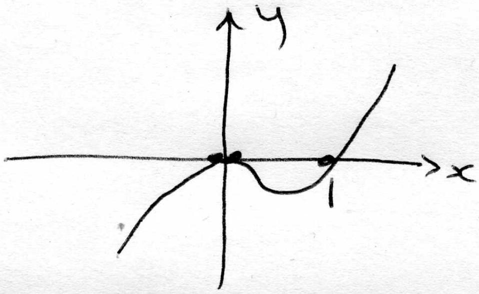
$\boxed{Q1}$ (a) $y = 1 - x^2 = (1-x)(1+x)$. Zeros @ $x = 1, -1$.

$y \to -\infty$ as $x \to \infty$.

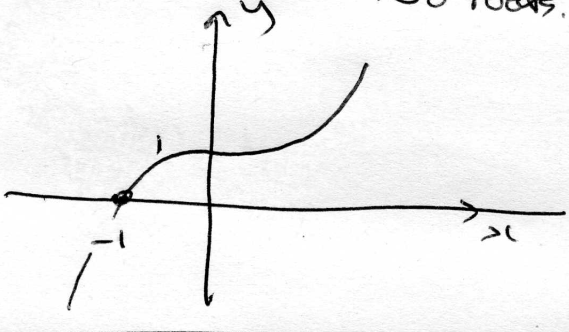


(b) $y = x^3 - x^2 = x^2(x-1)$. Zeros @ $x = 0, 0, 1$.

$y \to \infty$ as $x \to \infty$.



(c) $y = x^3 + 1 = (x-1)\underbrace{(x^2 + x + 1)}_{\text{No roots.}}$

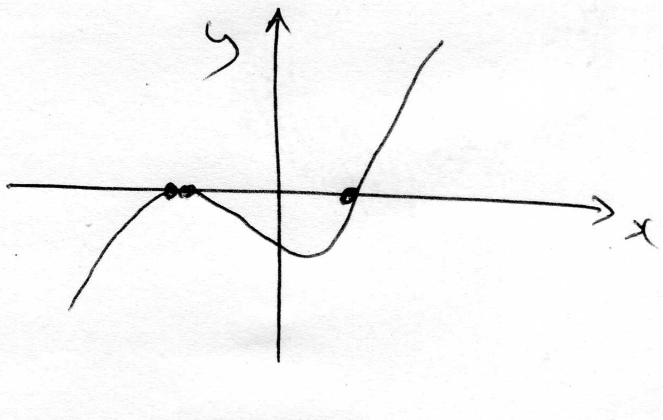


Inflexion pt. at $x = 0$ because this is $y = x^3$ raised up by 1.

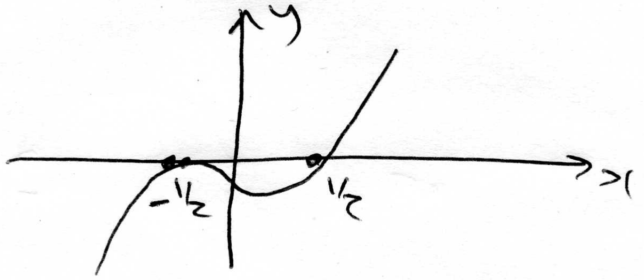
(d) $y = x^3 + x^2 - x - 1 = (x^2-1)(x+1) = (x-1)(x+1)^2$

Zeros at $x = 1, -1, -1$.

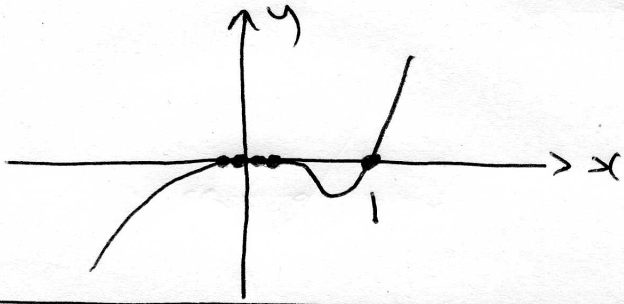
$y \to \infty$ as $x \to \infty$



(e) This is the same as Q1d if x in Q1d is replaced by $2x$. So $y = (2x-1)(2x+1)^2$

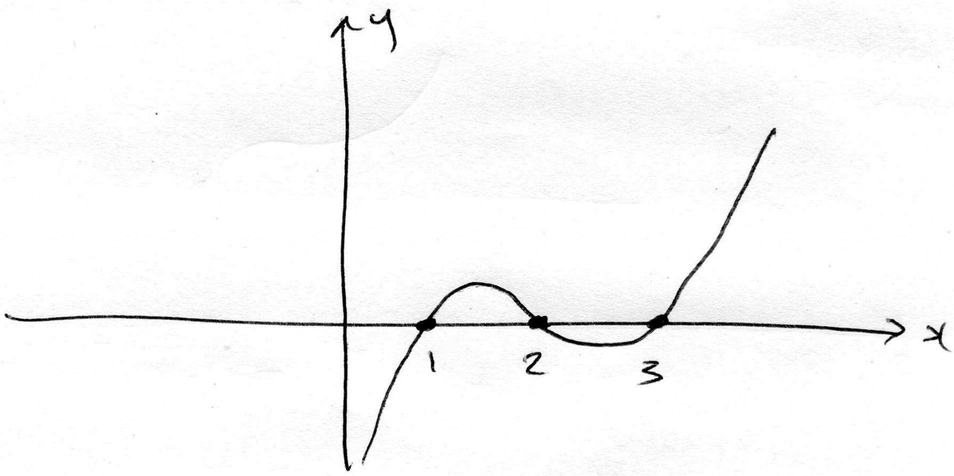


(f) $y = x^5 - x^4 = x^4(x-1)$. Zeros at $x = 1, 0, 0, 0, 0$.

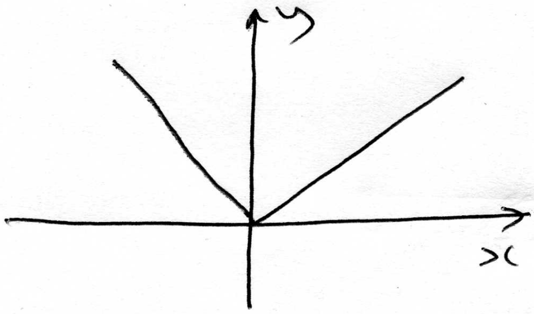


(g) $y = x^3 - 6x^2 + 11x - 6 = (x-1)(x^2-5x+6)$ [$x=1$ works]

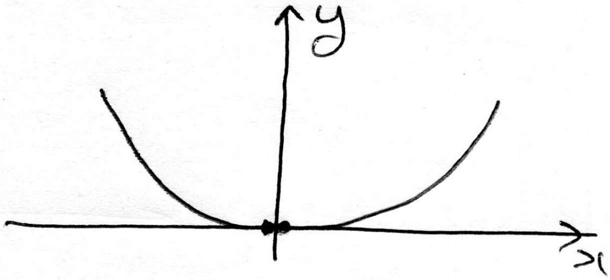
$= (x-1)(x-2)(x-3)$



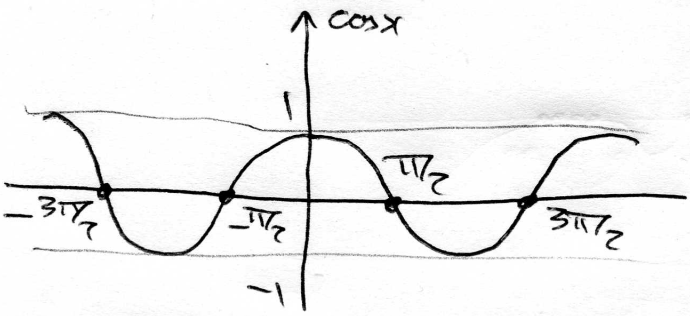
Q2 (a) $y = |x|$

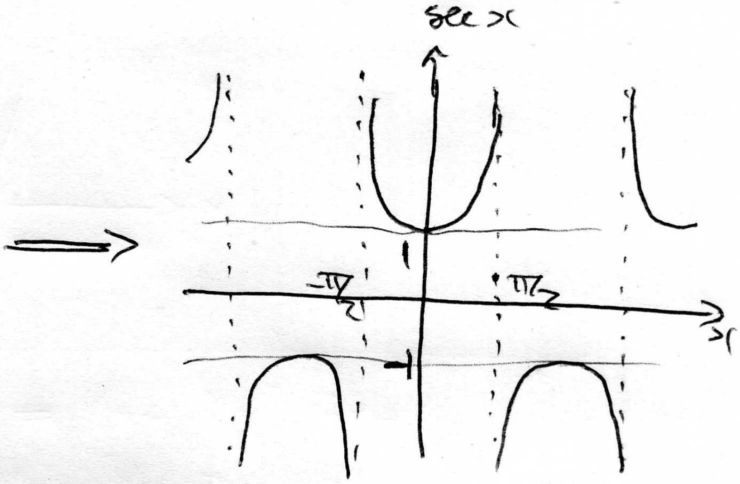


(b) $y = |x^2| = x^2$

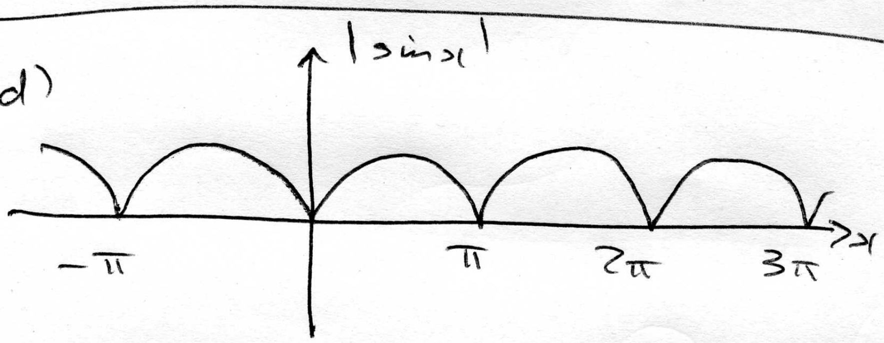


(c) $y = \sec x = \dfrac{1}{\cos x}$

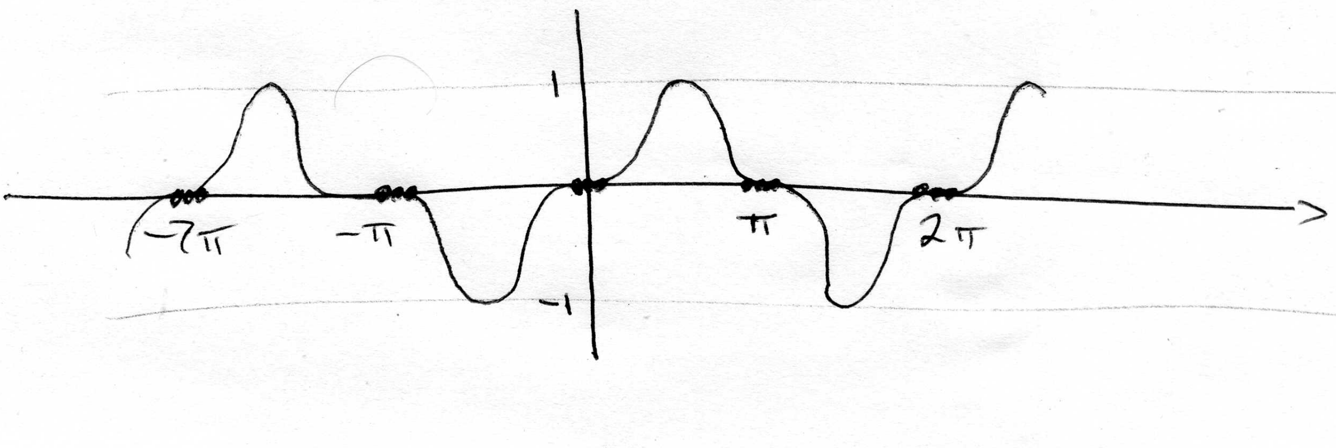




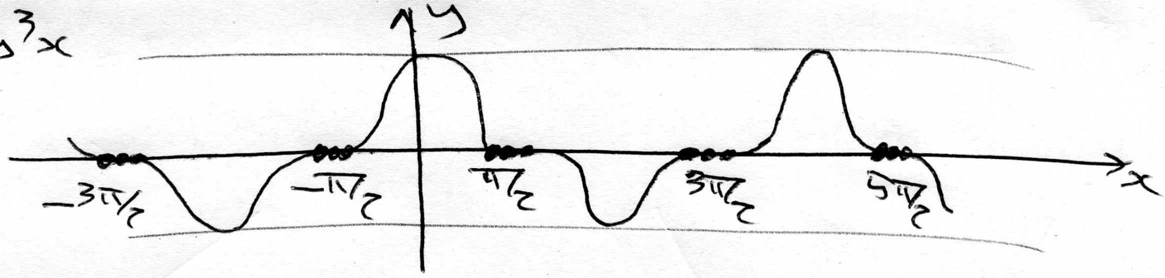
(d)



(e) $y = \sin^3 x$ <u>Triple</u> zeros at $x = 0, \pm\pi, \pm 2\pi, \ldots$

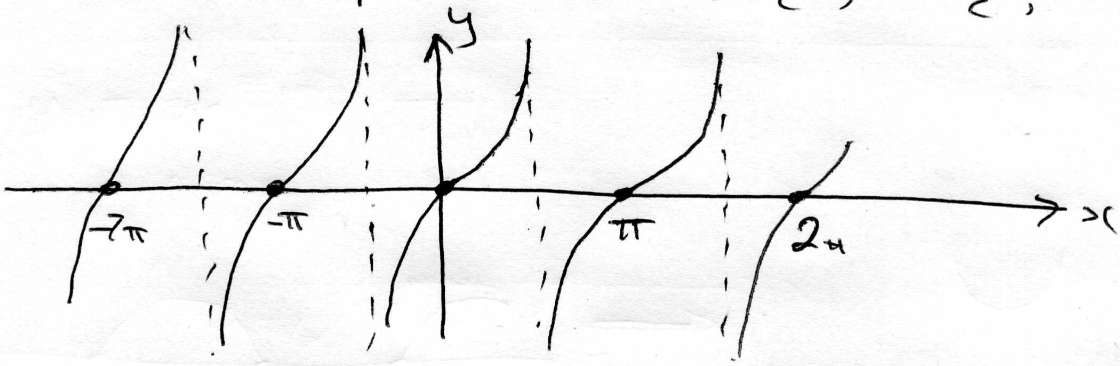


(f) $y = \cos^3 x$



(g) $y = \tan x$ zeros at $x = 0, \pm\pi, \pm 2\pi, \ldots$

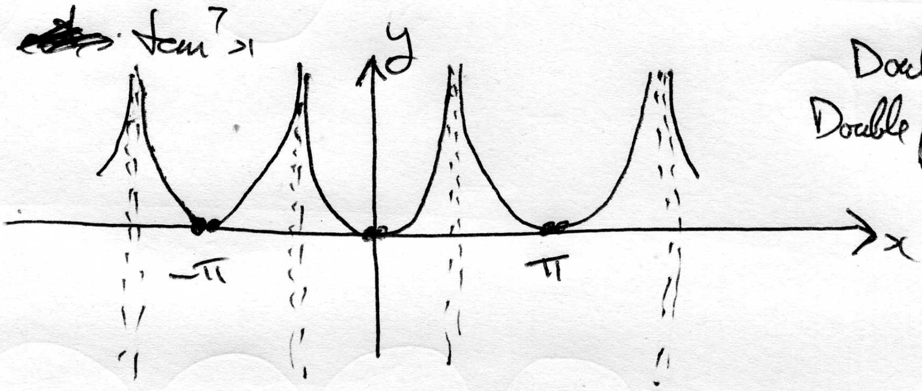
poles at $x = \pm \pi/2, \pm 3\pi/2, \ldots$



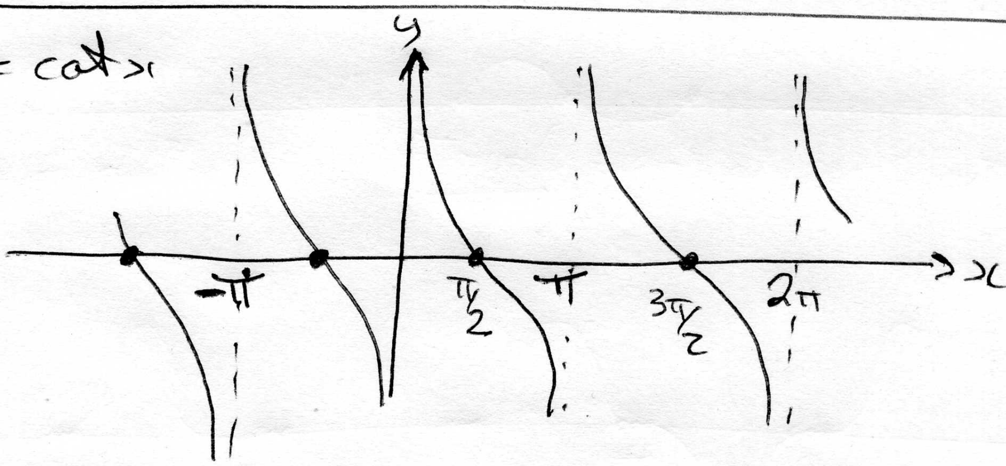
(h) $y = \tan^2 x$

Double zeros at $x = 0, \pm\pi$

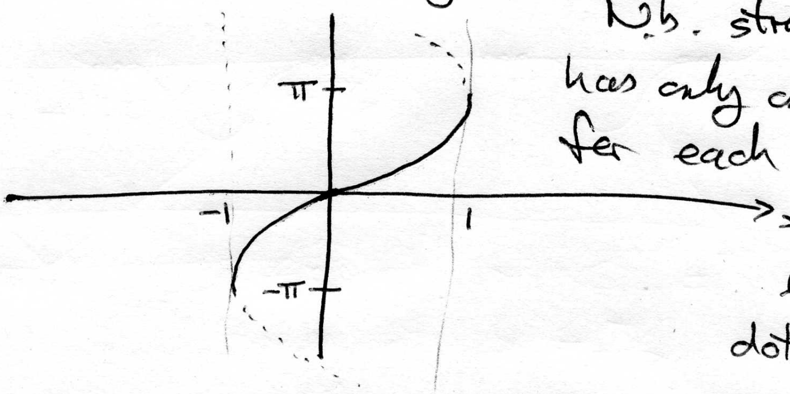
Double poles at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ ..



(i) $y = \cot x$

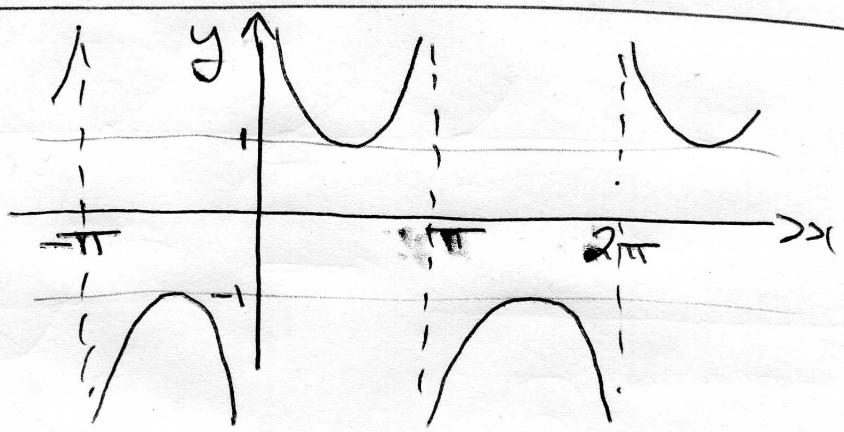


(j) $y = \sin^{-1} x \implies x = \sin y$

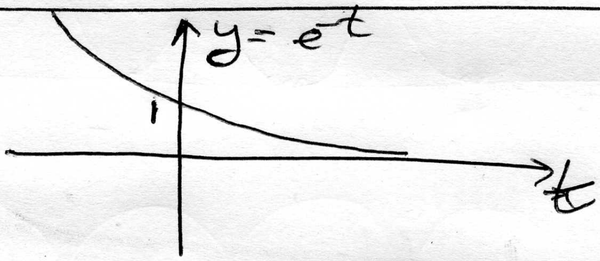


N.b. strictly, a function has only one value defined for each value of x. This is the continuous line, but <u>not</u> the dotted line.

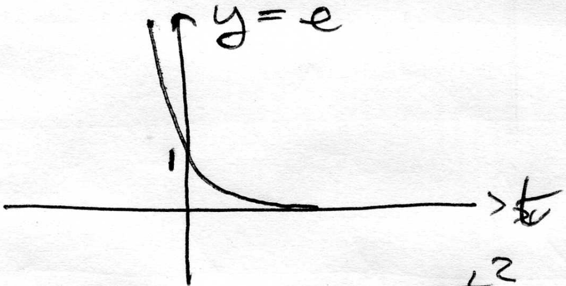
(k) $y = (\sin x)^{-1}$
$= \dfrac{1}{\sin x}$



$\boxed{3}$ (a) $y = e^{-t}$

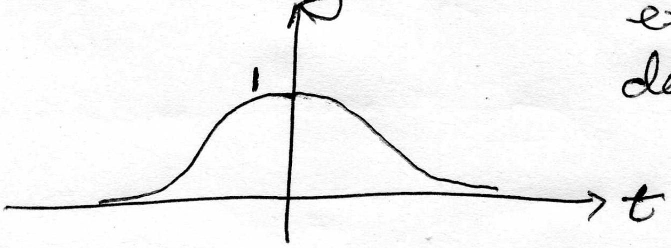


(b) $y = e^{-25t}$



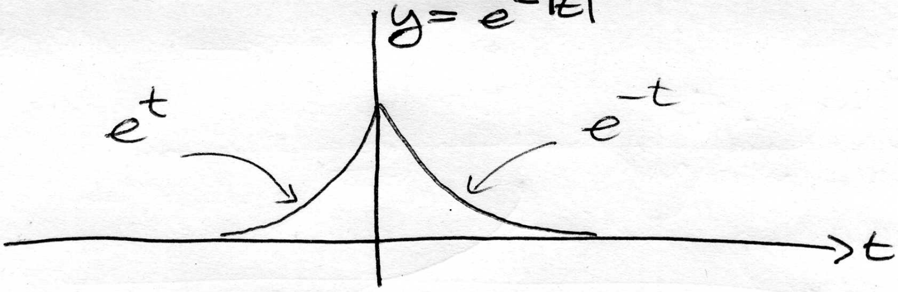
As Q3a but decays much more rapidly.

(c) $y = e^{-t^2}$

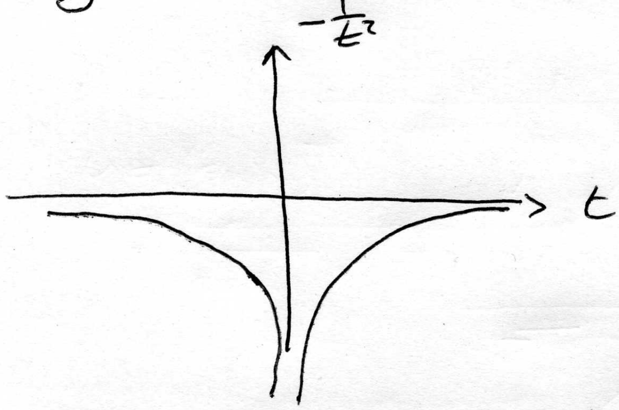


This function exhibits super exponential decay.

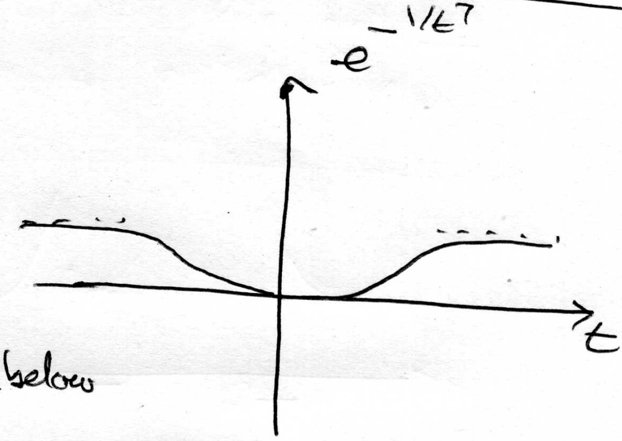
(d)



(e) $y = e^{-1/t^2}$



$\Longrightarrow$

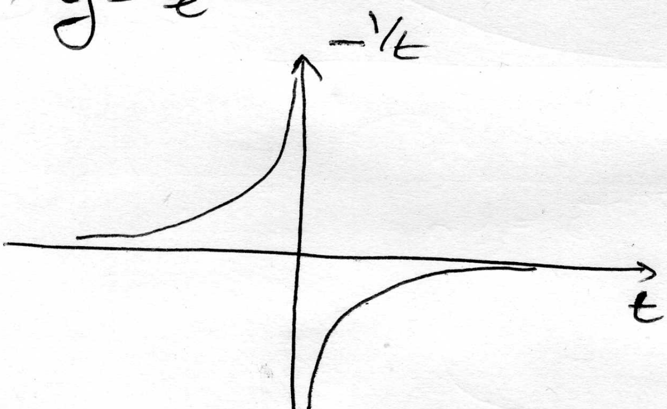


from below

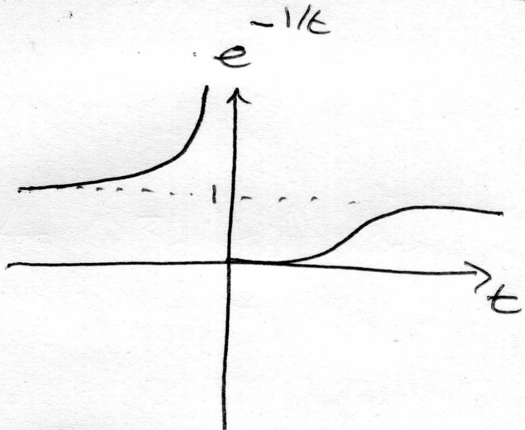
<u>Or:</u> $t \to \infty \Rightarrow -\frac{1}{t^2} \longrightarrow 0^- \Rightarrow e^{-1/t^2} \longrightarrow 1^-$

$t \to 0 \Rightarrow -\frac{1}{t^2} \longrightarrow -\infty \Rightarrow e^{-1/t^2} \longrightarrow 0.$

(f) $y = e^{-1/t}$



$\Longrightarrow$



<u>Or</u>

$t \to \infty \Rightarrow -1/t \to 0^- \Rightarrow e^{-1/t} \longrightarrow 1^-$

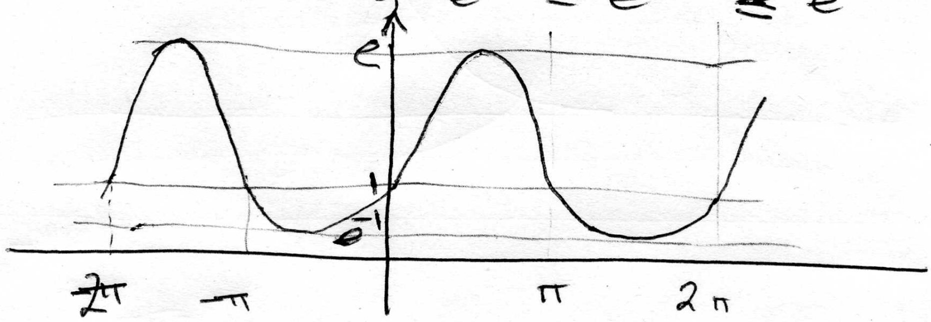
$t \to 0^+$ " $\longrightarrow -\infty$ " $\longrightarrow 0$

$t \to 0^-$ " $\longrightarrow +\infty$ " $\longrightarrow \infty$

$t \to -\infty$ " $\longrightarrow 0^+$ " $\longrightarrow 1^+$

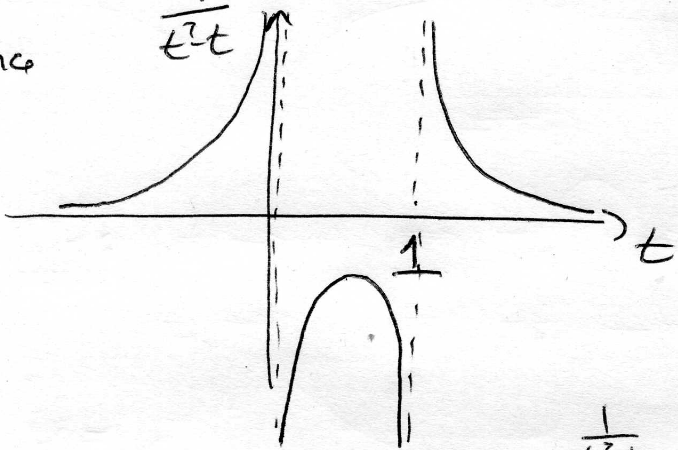
(g) $y = e^{\sin t}$. Given that $-1 \le \sin t \le 1$ then

$e^{-1} \le e^{\sin t} \le e^{1}$

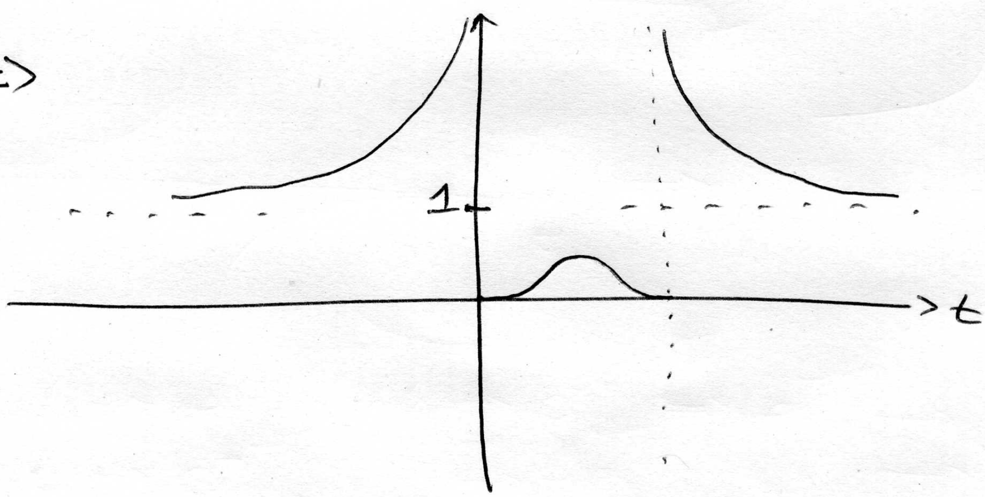


(h) $y = e^{1/(t^2-t)}$ Now $t^2 - t = t(t-1)$.

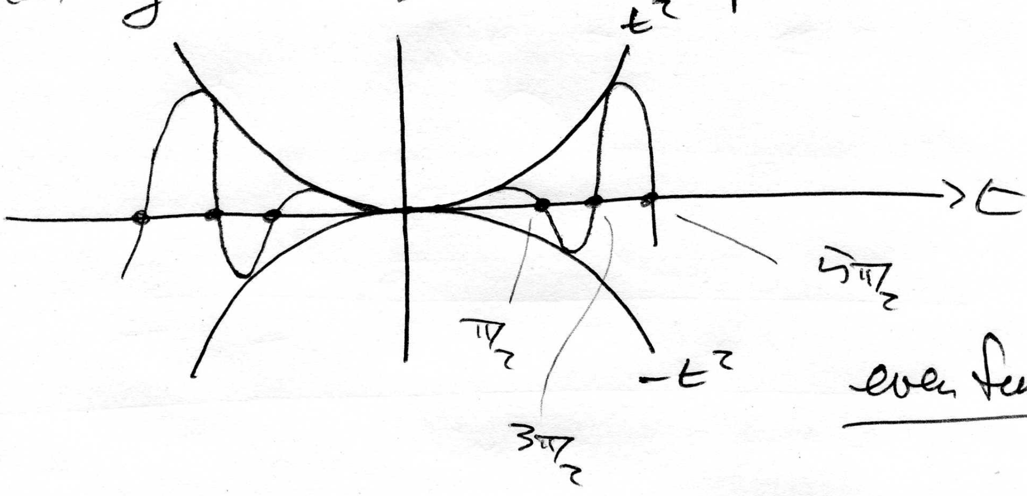
Hence $\frac{1}{t^2-t}$ poles @ $t = 0, 1$



$\Longrightarrow$ $e^{\frac{1}{t^2-t}}$



$\boxed{4}$ (a) $y = t^2 \cos t$. Envelopes: $-t^2$, $+t^2$.



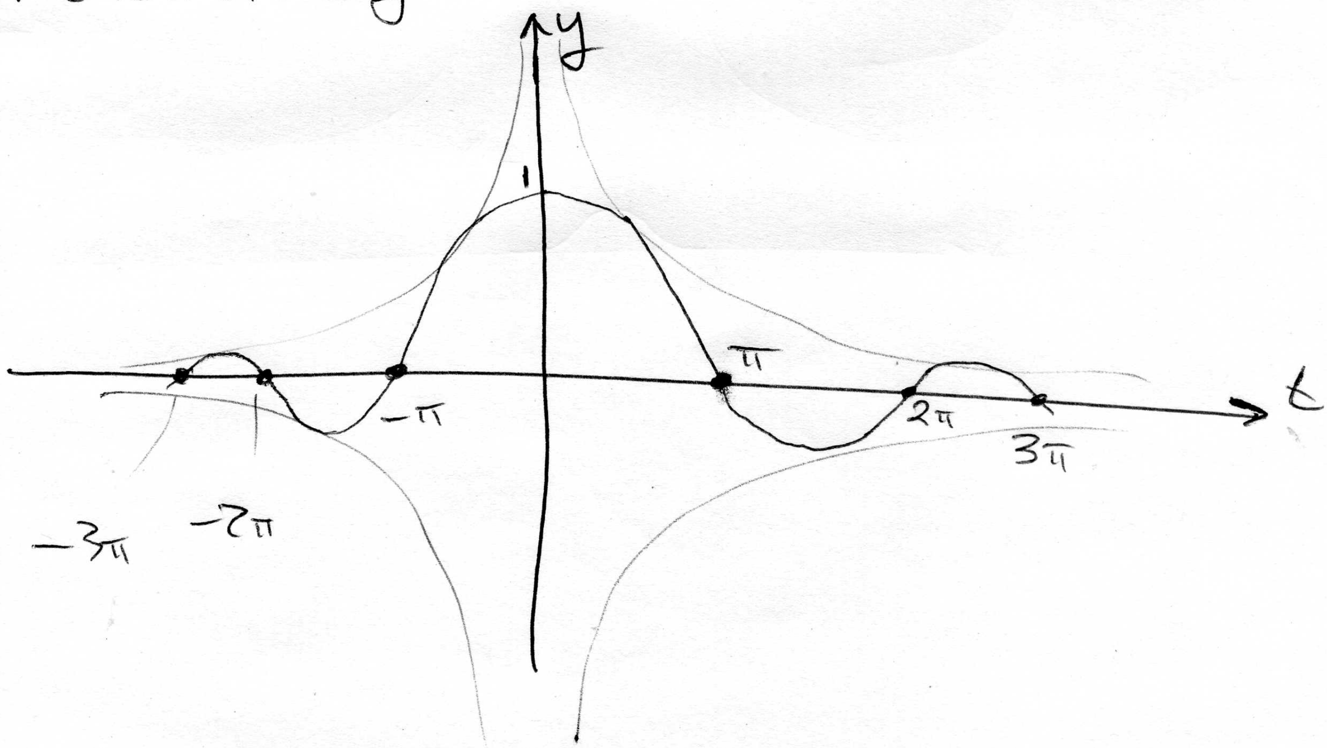
<u>even function</u>.

(b) $y = \dfrac{\sin t}{t}$. Envelopes: $\frac{1}{t}$, $-1/t$

Clearly zeros at $t = \pm\pi, \pm 2\pi, \ldots$

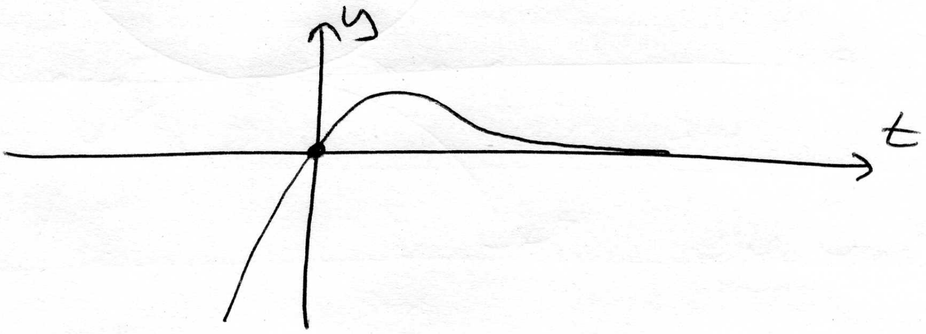
However, at $t=0$ we have a $\frac{0}{0}$ case. Later in the unit we'll use l'Hôpital's rule to show that $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Alternatively the letting of $t = 0.1, 0.01, 0.001$ etc [calculator set to RADIANS] yields the limit.



(c) $y = te^{-t}$.

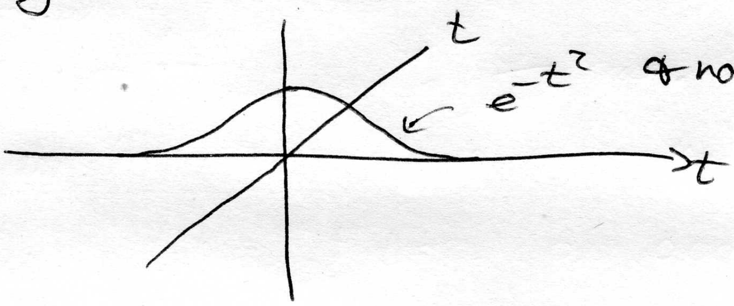
$y = 0$ @ $t = 0$.
$y \to 0$ as $t \to \infty$
$y \to -\infty$ as $t \to -\infty$



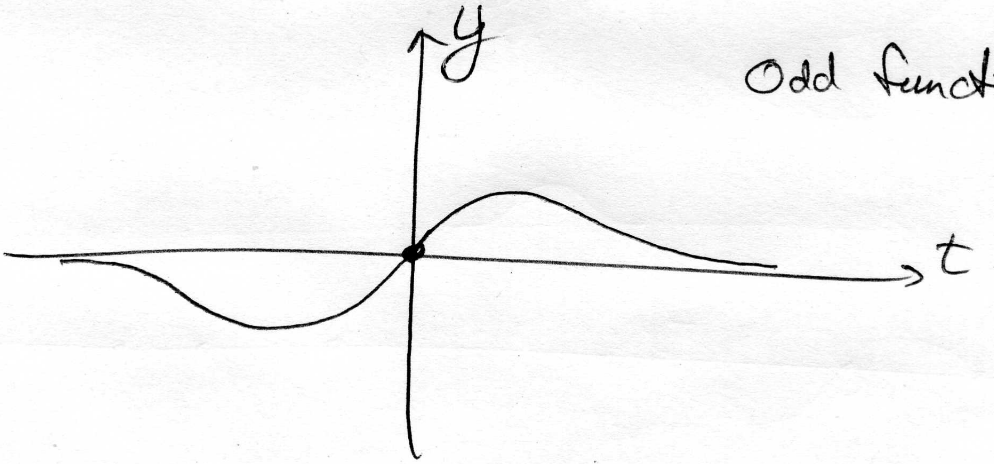
(d) $y = te^{-t^2}$.

Either: $y = 0$ at $t = 0$
$y \to 0$ as $t \to \pm\infty$

Or use:



& now multiply



Odd function.

These two subquestions aren't envelope questions!

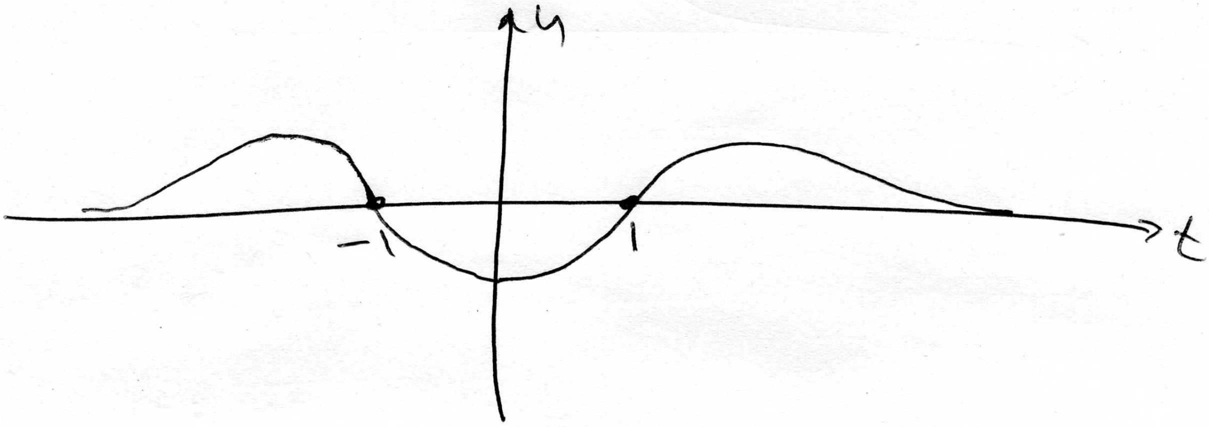
(e) $y = (t^2-1)e^{-t^2}$.

Zeros at $t = \pm 1$

$y \to 0$ as $t \to \pm\infty$.

$y(0) = -1$

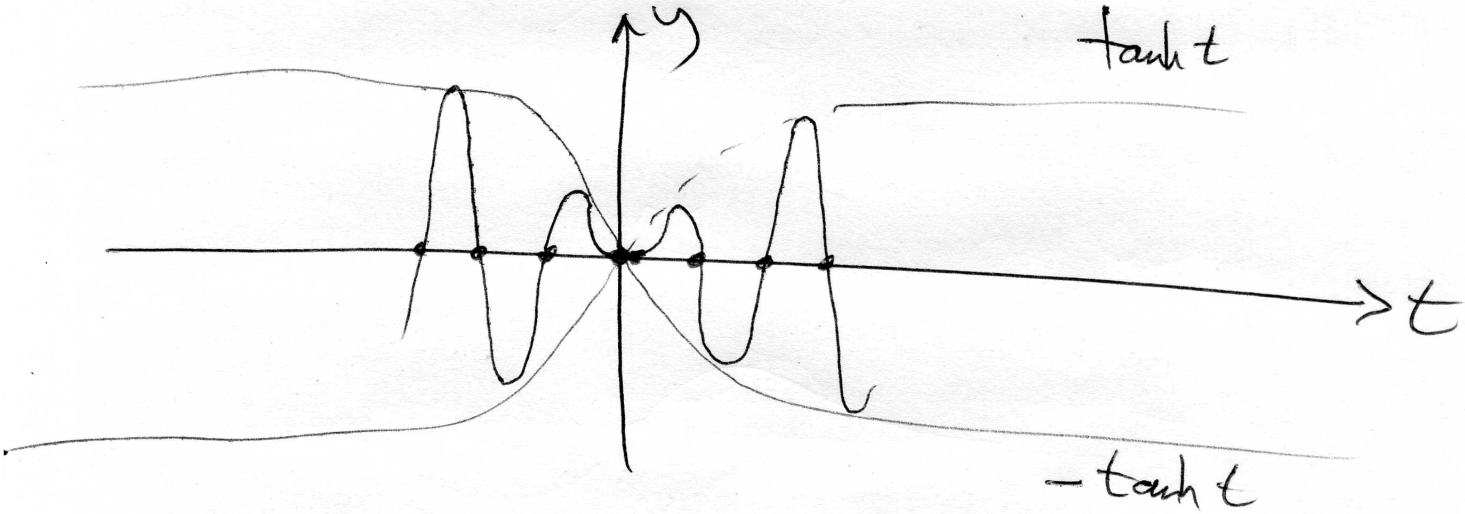
even function.



(f) $y = \underbrace{\tanh t}_{\substack{\text{zero} \\ \text{at } t=0}} \; \underbrace{\sin 100t}_{\substack{\text{zeros at} \\ t = \frac{n\pi}{100}}}$

Even function.

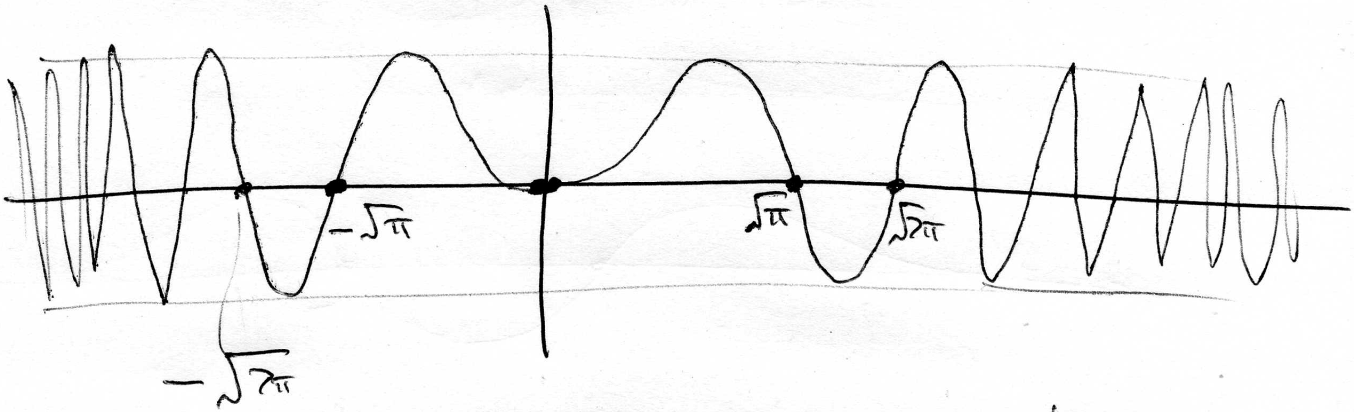
Double zero at $t=0$



(g) $y = \sin(t^2)$. Clearly $-1 \le y \le 1$.

Zeros when $t^2 = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$

or $t = \pm\sqrt{n\pi}$



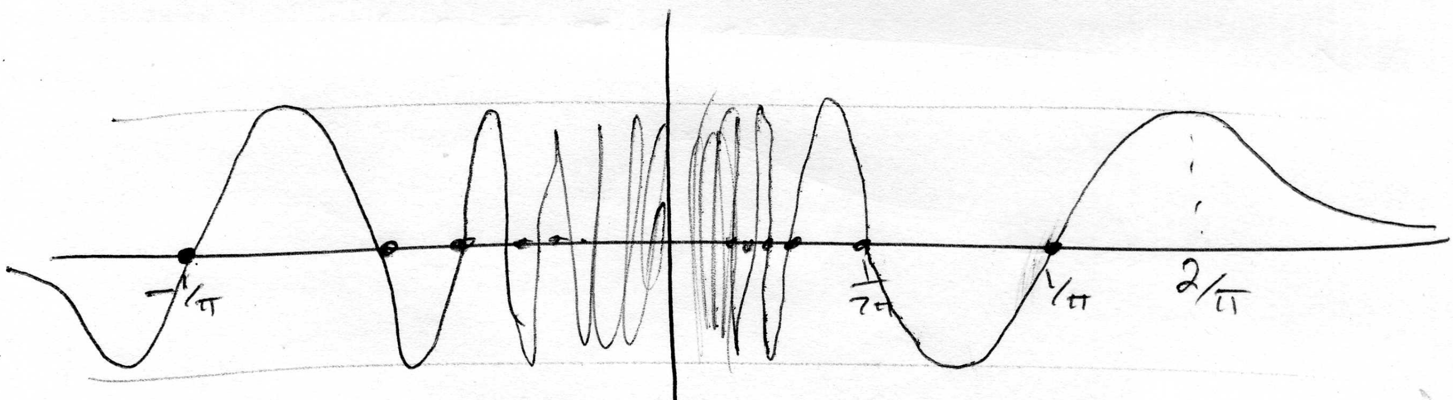
Sometimes known as a chirp signal. Used to find resonant frequencies of structures – usually aircraft.

Even function

(h) $y = \sin(t^{-1})$ Odd function. $-1 \le y \le 1$.

Zeros when $t^{-1} = n\pi \Rightarrow t = \frac{1}{n\pi}$ $(n = \pm 1, \pm 2, \ldots)$.

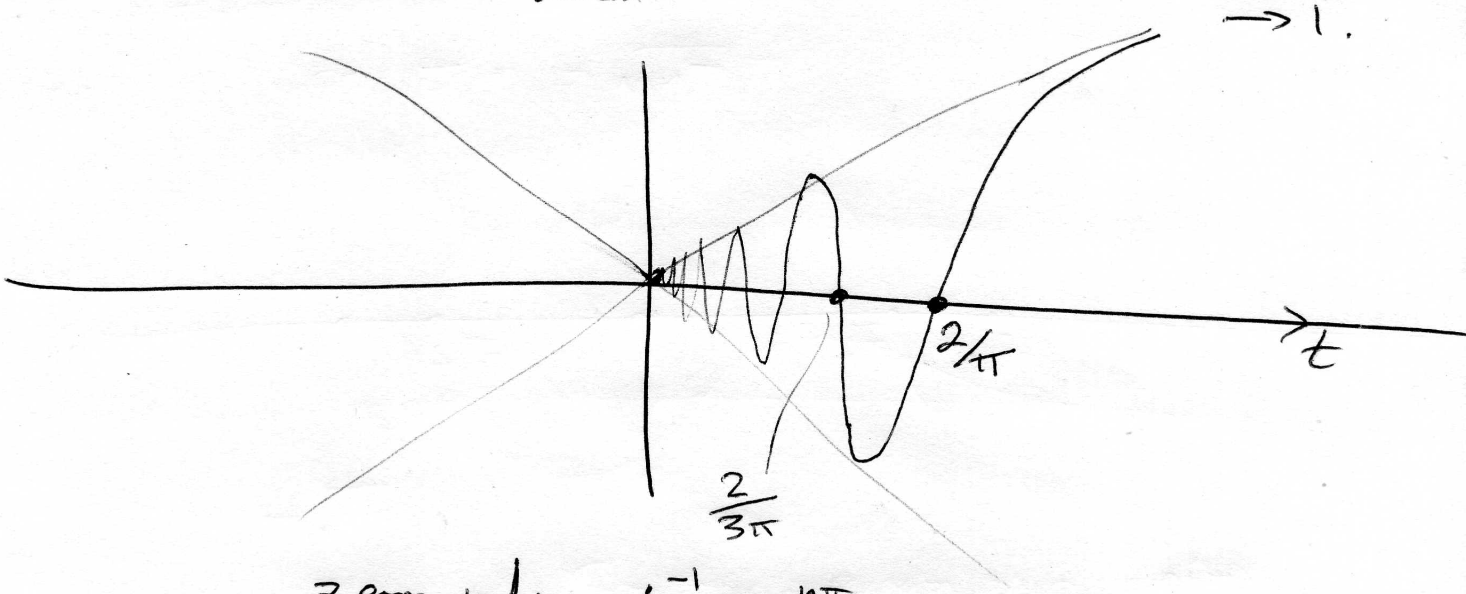
Largest t which is a zero is $1/\pi$.



$t \to \infty \Rightarrow t^{-1} \to 0^+ \Rightarrow \sin(t^{-1}) \to 0^+$.

(i) $y = t\cos(t^{-1})$ y is odd.

When $t \to \infty$, $t^{-1} \to 0^+ \Rightarrow \cos(t^{-1}) \to 1$.



zeros when $t^{-1} = \frac{n\pi}{2}$, $n = \pm 1, \pm 3, \pm 5 \ldots$

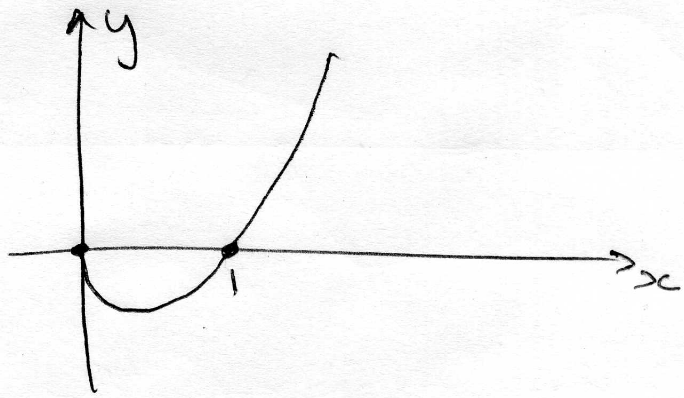
$\boxed{5}$ (a) = $y = x \ln x$

Function doesn't exist for $x < 0$.

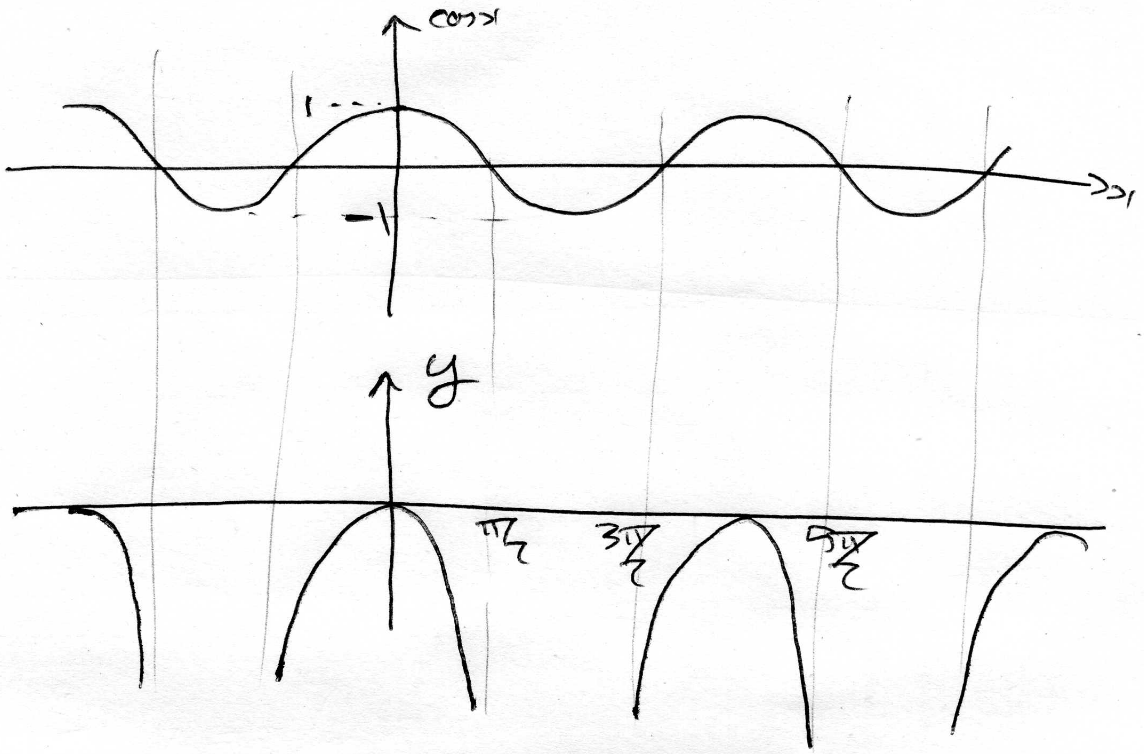
$y(1) = 0$.

Although $\ln x \to -\infty$ as $x \to 0^+$, $x\ln x \to 0$.

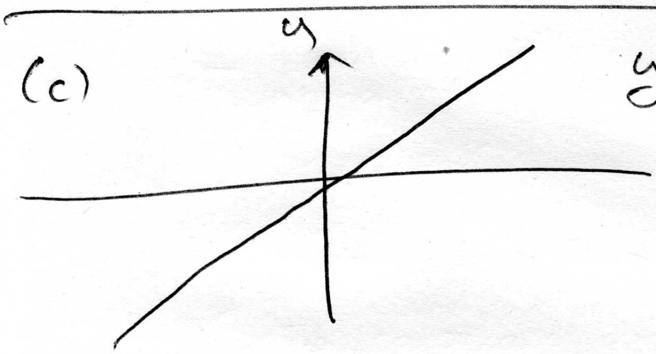
Some calculator work will confirm this, but again l'Hôpital's rule may be used.



(b) $y = \ln \cos x$. Nb. the function doesn't exist when $\cos x \leq 0$.

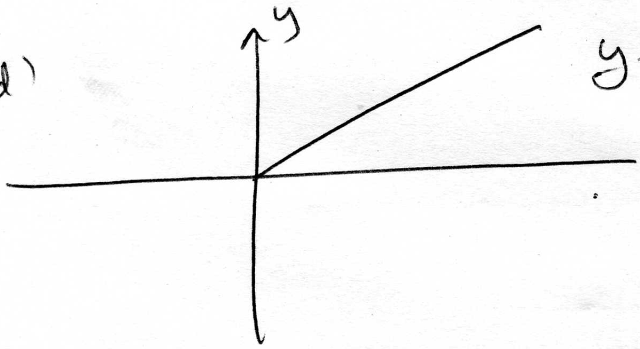


(c)



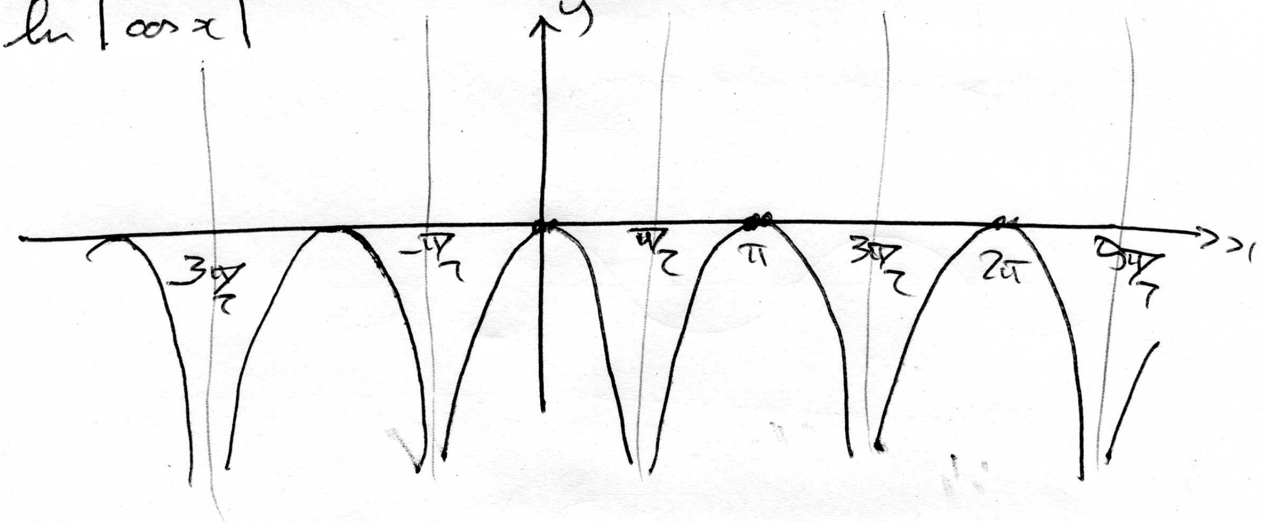
$y = \ln(e^x)$
$= x$

(d)



$y = e^{\ln x}$ [$\ln x$ not real when $x < 0$]

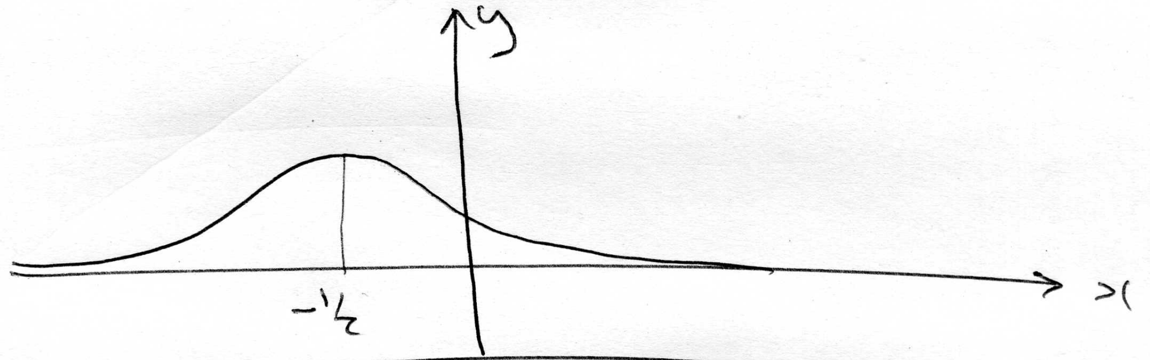
(e) $y = \ln|\cos x|$



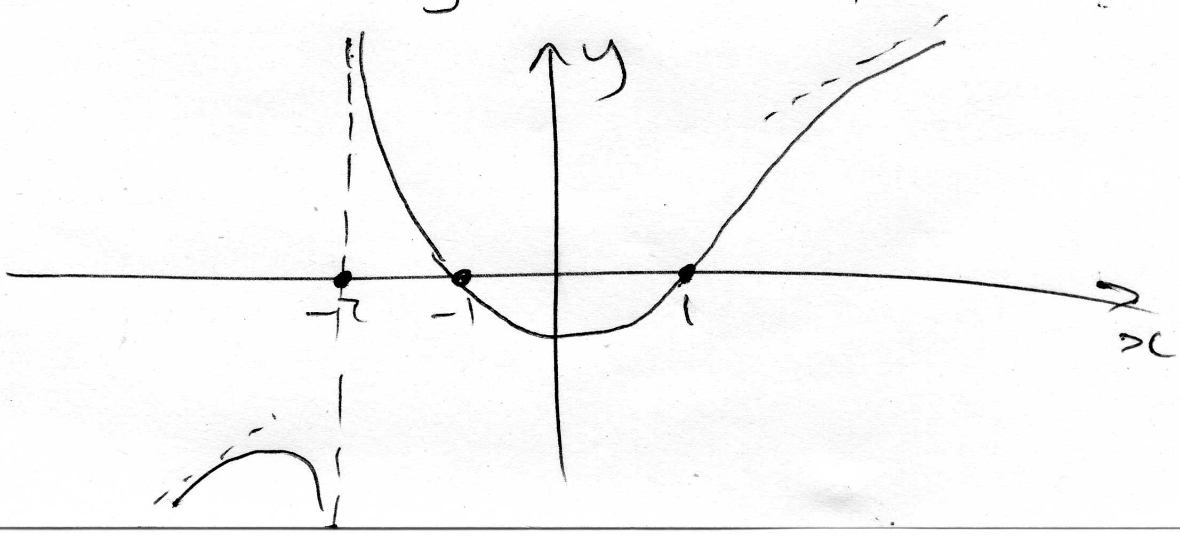
6 (a) $y = (x^2+x+1)^{-1}$

But $x^2+x+1 = (x+\tfrac{1}{2})^2 + \tfrac{3}{4} > 0$

$\Longrightarrow$ no poles, but y is a maximum when $x = -\tfrac{1}{2}$.



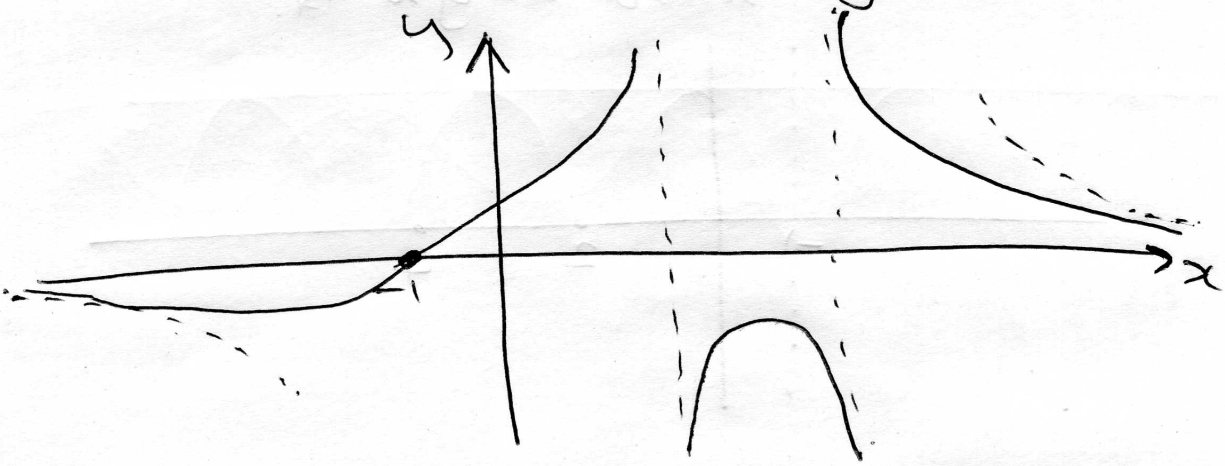
(b) $y = \dfrac{x^2-1}{x+2}$ $\Longrightarrow$ Zeros at $x=-1,1$
Pole at $x=-2$
$y \sim x$ when $|x| \gg 1$



(c) $y = \dfrac{x+1}{(x-2)(x-4)} \Longrightarrow$ zero @ $x = -1$

poles @ $x = 2, 4$

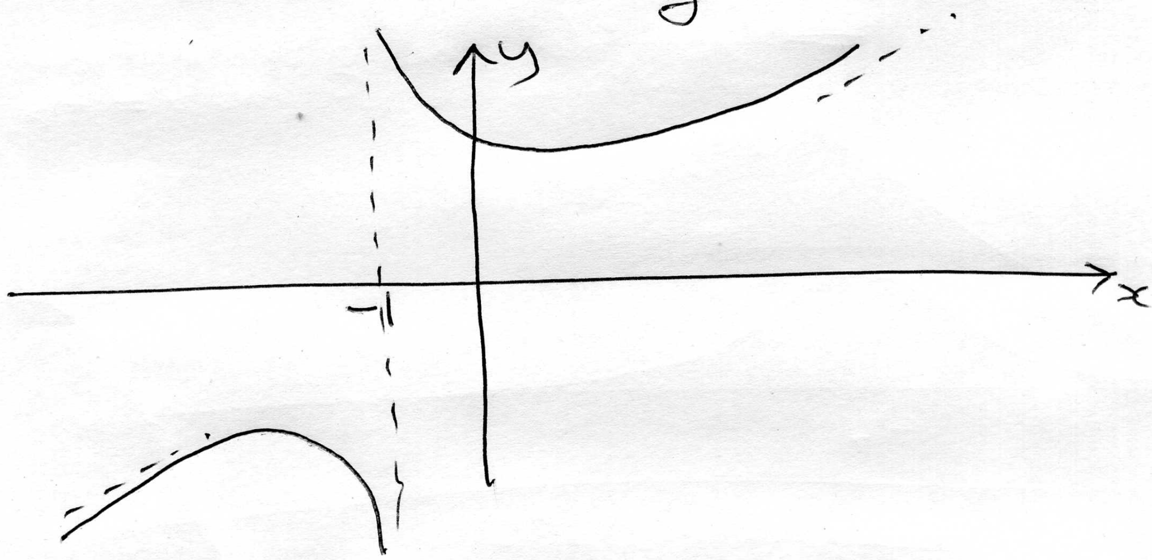
$x \to \infty \Rightarrow y \sim 1/x$.



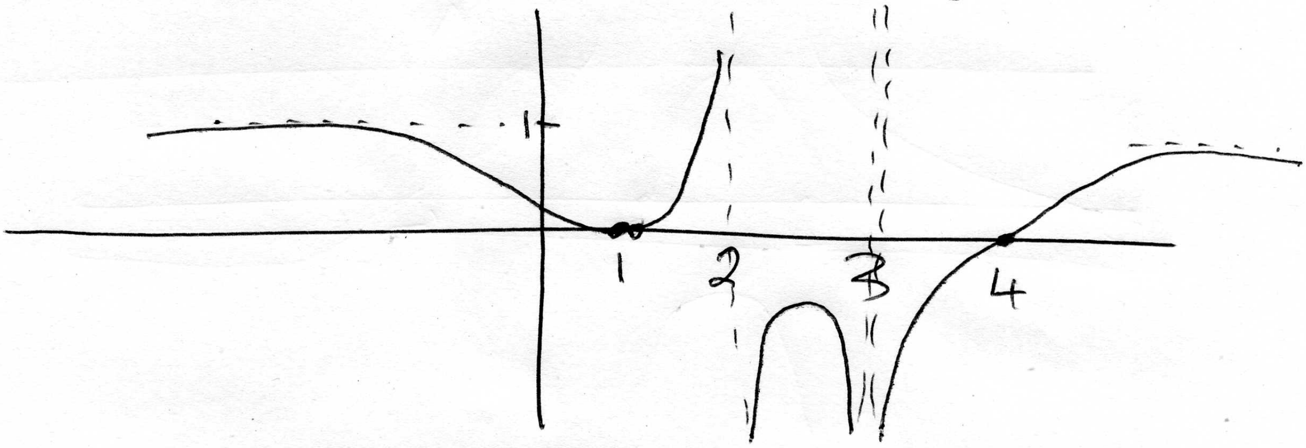
(d) $y = \dfrac{x^2+1}{x+1} \Longrightarrow$ No zeros

Pole at $x = -1$

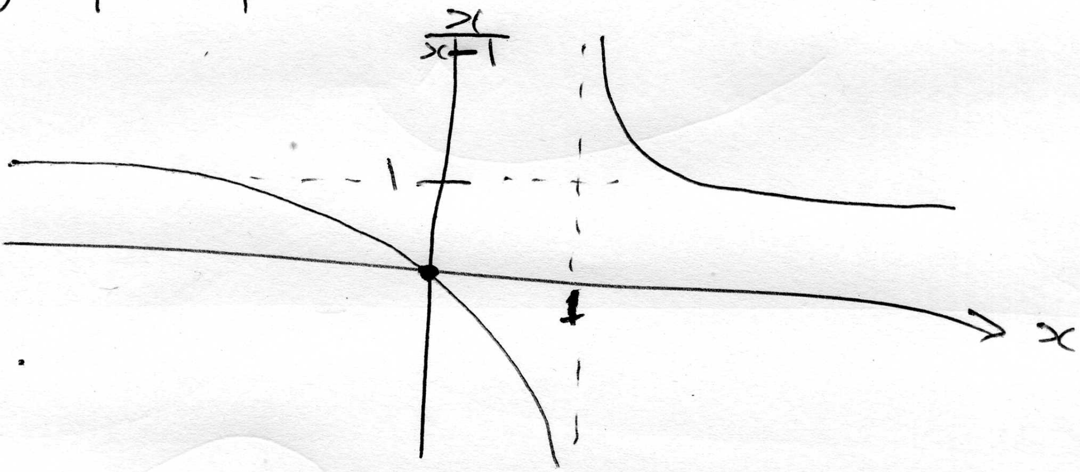
$x \to \infty \Rightarrow y \sim x$.



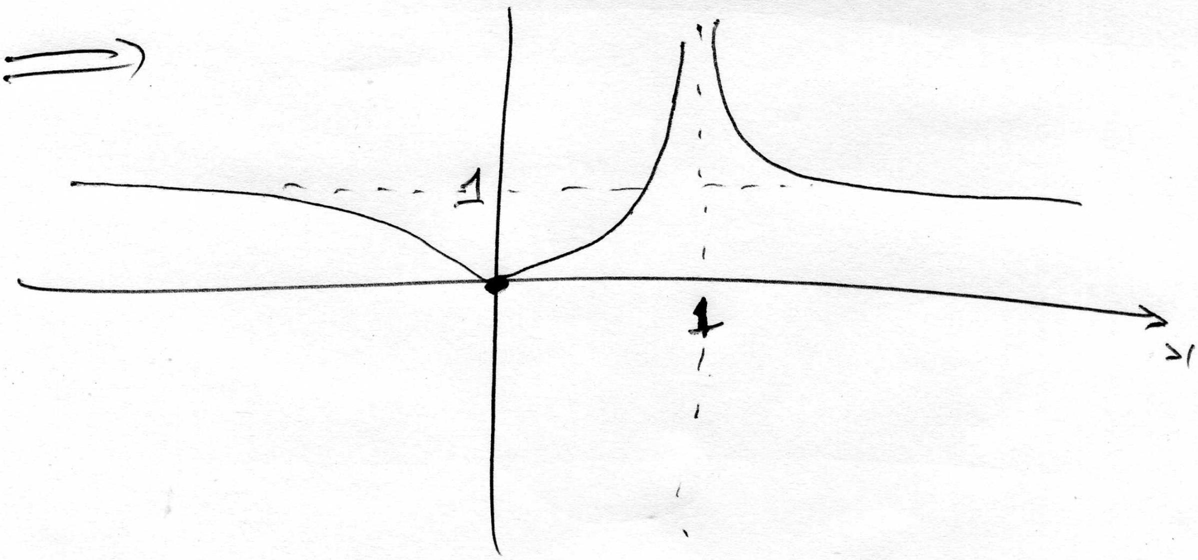
(e) $y = \dfrac{(x-1)^2(x-4)}{(x-2)(x-3)^2} \Longrightarrow \begin{cases} \text{Zeros @ } x = 1, 1, 4 \\ \text{Poles @ } x = 2, 3, 3 \\ x \to \infty \Rightarrow y \to 1. \end{cases}$



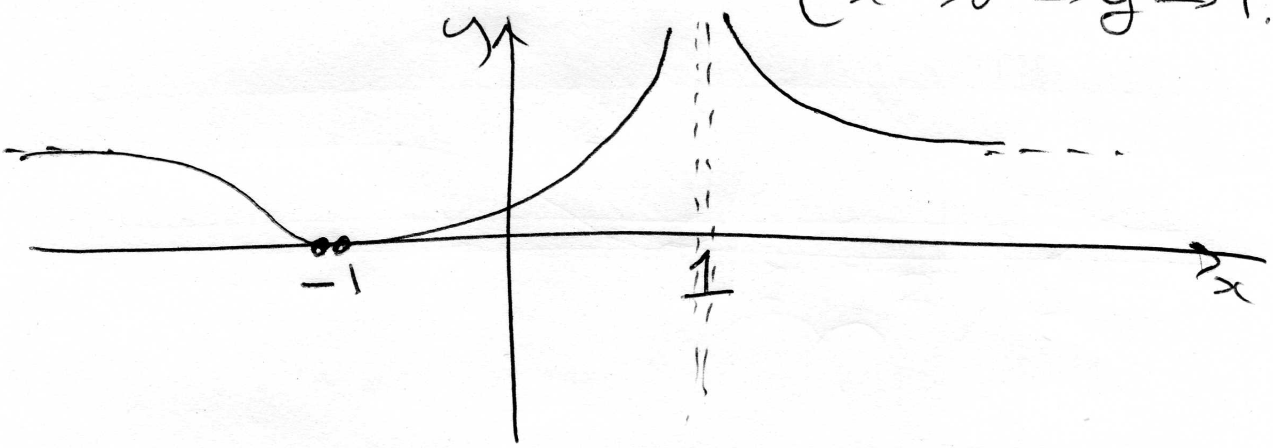
(f) $y = \left|\dfrac{x}{x-1}\right|$. Could sketch $\dfrac{x}{x-1}$ first:



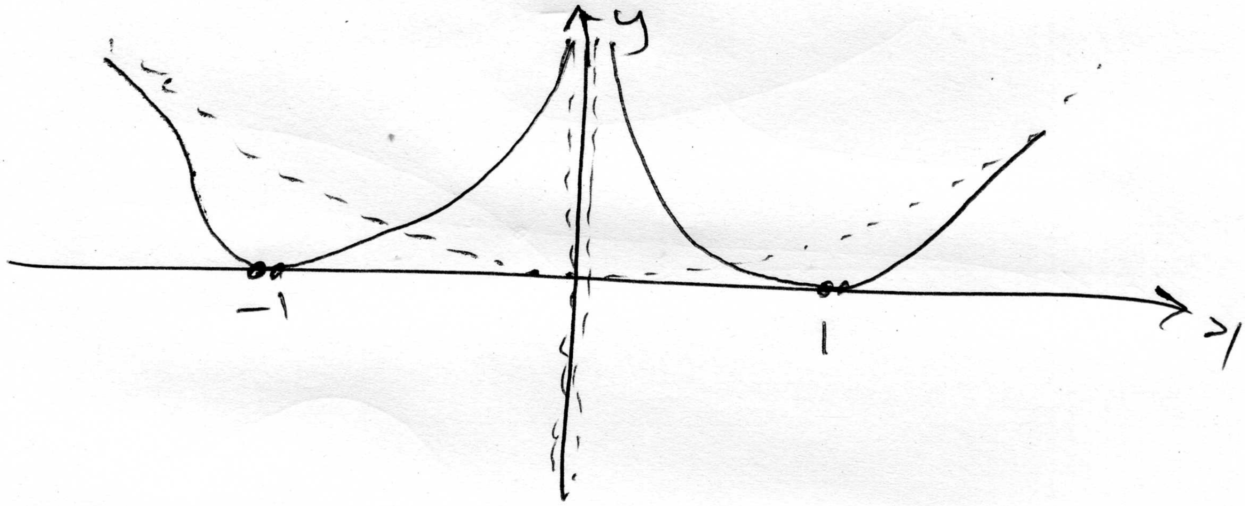
$\Longrightarrow$



(g) $$y = \frac{x^2+2x+1}{x^2-2x+1} = \frac{(x+1)^2}{(x-1)^2} \Rightarrow \begin{cases} \text{Zeros @ } x = -1, -1 \\ \text{Poles @ } x = 1, 1 \\ x \to \infty \Rightarrow y \to 1. \end{cases}$$



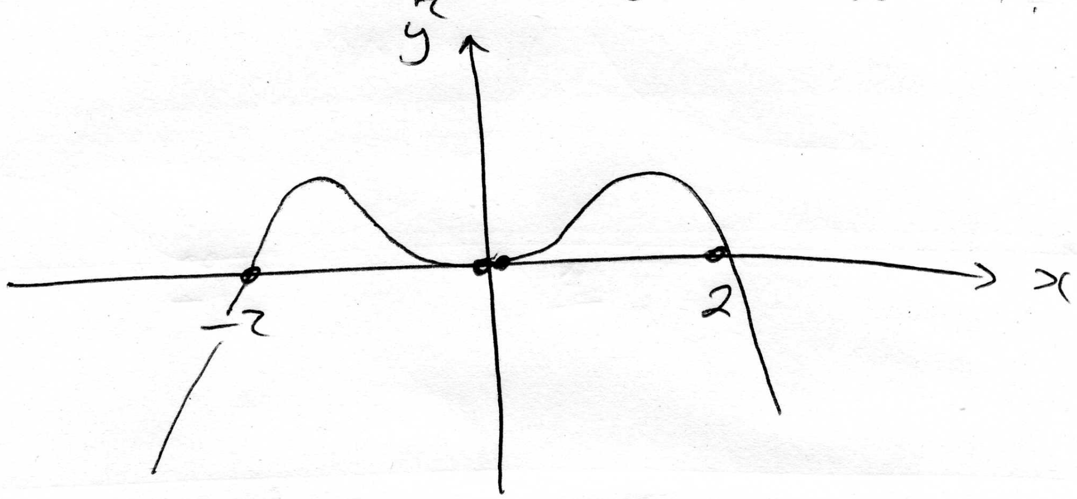
(h) $$y = \frac{(x^2-1)^2}{x^2} \Rightarrow \begin{cases} \text{Zeros @ } x = -1, -1, 1, 1 \\ \text{Poles @ } x = 0, 0 \\ y \approx x^2 \text{ when } |x| \gg 1. \end{cases}$$



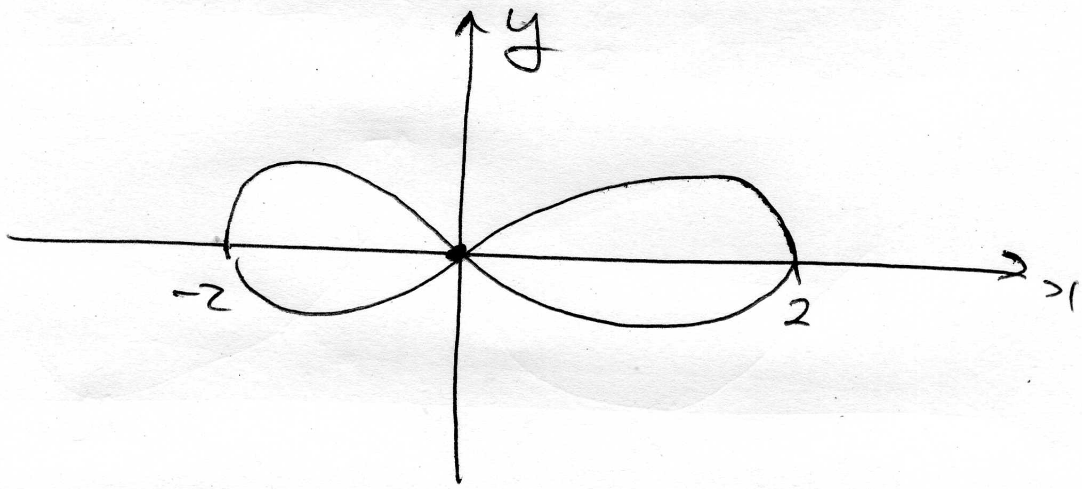
$\boxed{7}$ (a) $y^2 = 4x^2 - x^4 = x^2(4 - x^2)$

$= x^2(2-x)(2+x)$.

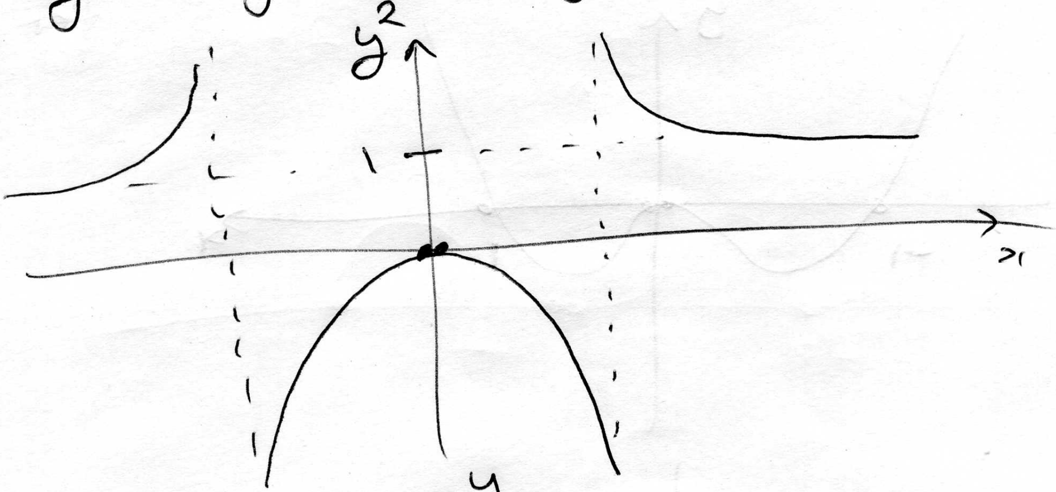
$\Longrightarrow$



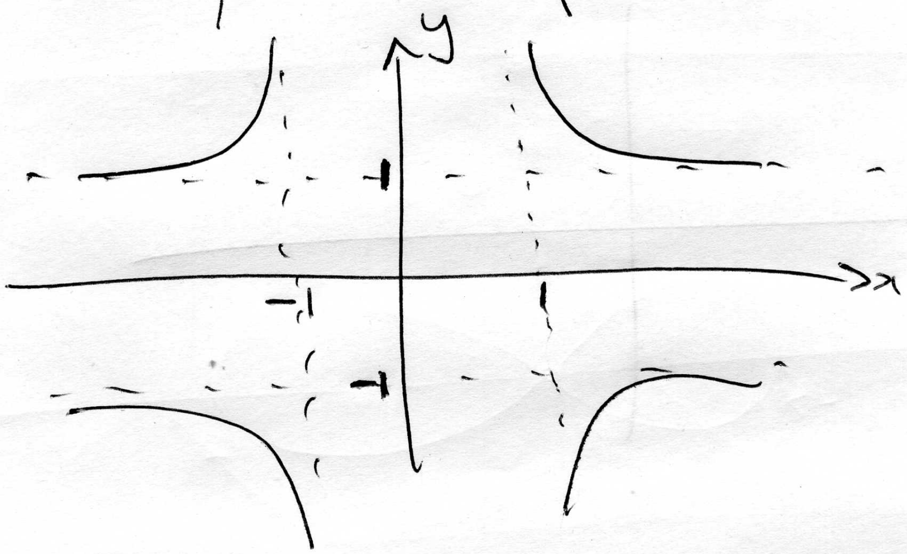
$\Longrightarrow$



(b) $x^2+y^2=x^2y^2 \implies y^2=\dfrac{x^2}{x^2-1}$



$\implies$



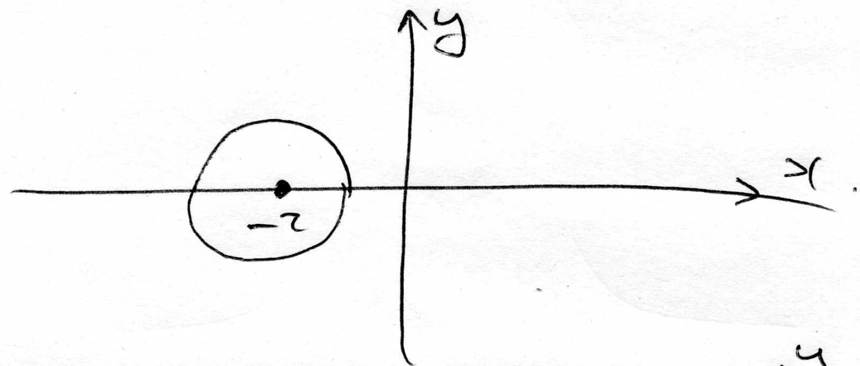
N.b. $x^2=\dfrac{y^2}{y^2-1}$ as well.

Hence the extra symmetry.

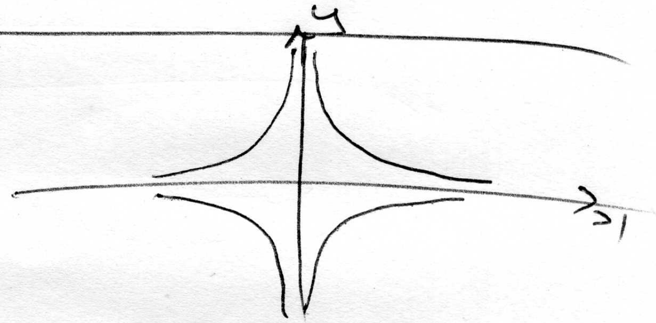
(c) $x^2 + 4x + y^2 = -3$

$\Rightarrow x^2 + 4x + 4 + y^2 = 1$ (completing the square)

$\Rightarrow (x+2)^2 + y^2 = 1 \Rightarrow$ circle of radius 1
centre $= (-2, 0)$.

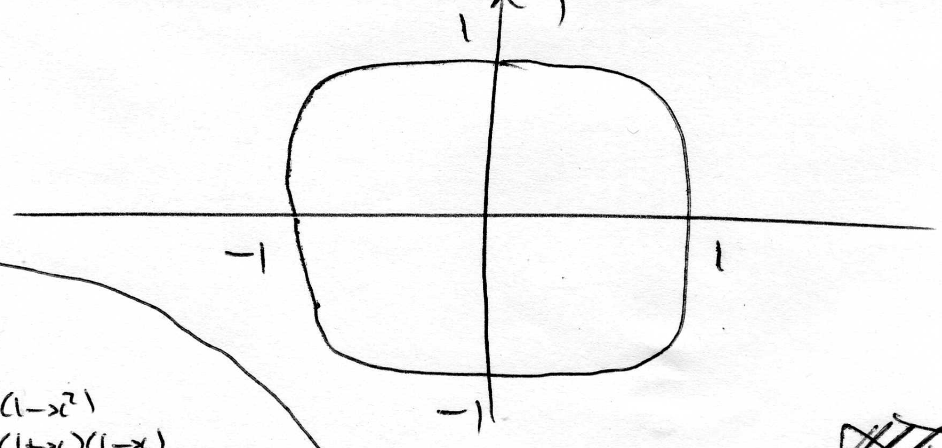


(d) $x^2 y^2 = 1 \Rightarrow y = \pm \frac{1}{x}$



(e) $y^4 + x^4 = 1$

This is a bloated circle:



Advanced:
$y^4 = 1 - x^4$
$= (1+x^2)(1-x^2)$
$= (1+x^2)(1+x)(1-x)$.
Near $x = 1$: $y^4 \approx 4(1-x)$
or $y = \pm\sqrt{2}(1-x)^{1/4}$
Quartic!!