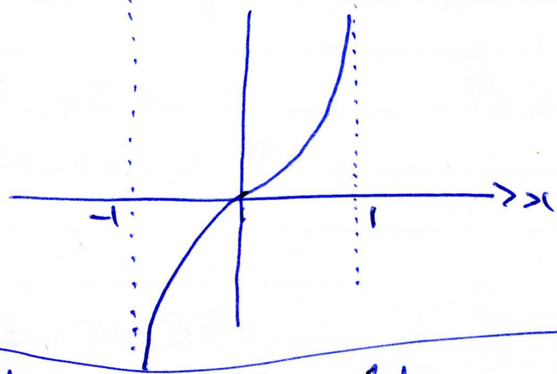
1. (a) $y = \frac{x}{(1-x^2)^{1/2}} \Rightarrow y' = \left[(1-x^2)^{1/2} \times 1 - \frac{x(-2x)}{2(1-x^2)^{1/2}} \right] / (1-x^2)$

$= \cancel{\frac{[1+x^2]}{[1-x^2]^{3/2}}} \quad \frac{1}{(1-x^2)^{3/2}}$ [29/12/2016]

(b) Function exists only when $x^2<1$. Has a zero at $x=0$. Also $y \to \infty$ as $x \to \pm 1$. Hence,



(c) $I = \int_0^1 \frac{x\,dx}{(1-x^2)^{1/2}}$ —— let $x = \sin\xi \Rightarrow dx = \cos\xi\, d\xi$

Also $x=0 \Rightarrow \xi = 0$, $x=1 \Rightarrow \xi = \pi/2$.

$= \int_0^{\pi/2} \frac{\sin\xi}{\cos\xi} \cos\xi\, d\xi = \int_0^{\pi/2} \sin\xi\, d\xi = 1$.

(d) $V = \pi \int_0^{1/2} y^2\, dx = \pi \int_0^{1/2} \frac{x^2}{1-x^2}\, dx = \pi \int_0^{1/2} \left[\frac{x^2-1+1}{1-x^2} \right] dx$

$= \pi \int_0^{1/2} \left[-1 + \frac{1}{1-x^2} \right] dx = \pi \int_0^{1/2} \left[-1 + \frac{1/2}{x+1} - \frac{1/2}{x-1} \right] dx$ using Partial Fractions

$= \pi \left[-x + \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| \right]_0^{1/2}$

$= \dots = \frac{\pi}{2}[\ln 3 - 1] \approx 0.1549$ (4DP)

(e) Should be $V = \pi \int_0^1 x^2\, dy$.

But $y = \frac{x}{(1-x^2)^{1/2}} \Rightarrow y^2 = \frac{x^2}{1-x^2} \Rightarrow (1-x^2)y^2 = x^2 \Rightarrow x^2 = \frac{y^2}{1+y^2}$

So $V = \pi \int_0^1 \frac{y^2}{1+y^2}\, dy = \pi \int_0^1 \left[\frac{y^2+1-1}{y^2+1} \right] dy = \pi \int_0^1 \left[1 - \frac{1}{1+y^2} \right] dy$

$= \pi \left[y - \tan^{-1} y \right]_0^1$ using standard result

$= \pi [1 - \pi/4] = 0.6742$ (4DP)

[2] (a) See Q7a on sheet 8 for the Binomial analysis for $(1+x)^{-1/2}$. If we change x to $-x$ all the way, then we get
$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!\, x^n}{2^{2n}(n!)^2}.$$

The same series is found using Taylor's series.

(b) Change x to x^2 in Q7a. Hence
$$(1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{2^{2n}(n!)^2}$$

(c) In (a) we have $u_n = \dfrac{(2n)!\, x^n}{2^{2n}\, n!\, n!}$.

Hence
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!\, x^{n+1}}{2^{2n+2}(n+1)!(n+1)!} \times \frac{2^{2n}\, n!\, n!}{(2n)!\, x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{4(n+1)(n+1)}\, x\right| = |x| \implies \text{Radius of convergence is } 1.$$

(d)
$$\sin^{-1} y = \int_0^y \frac{dx}{(1-x^2)^{1/2}} = \int_0^y \left[\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{2^{2n}(n!)^2}\right] dx$$
$$= \sum_{n=0}^{\infty} \frac{(2n)!\, y^{2n+1}}{(2n+1)\, 2^{2n}(n!)^2}$$

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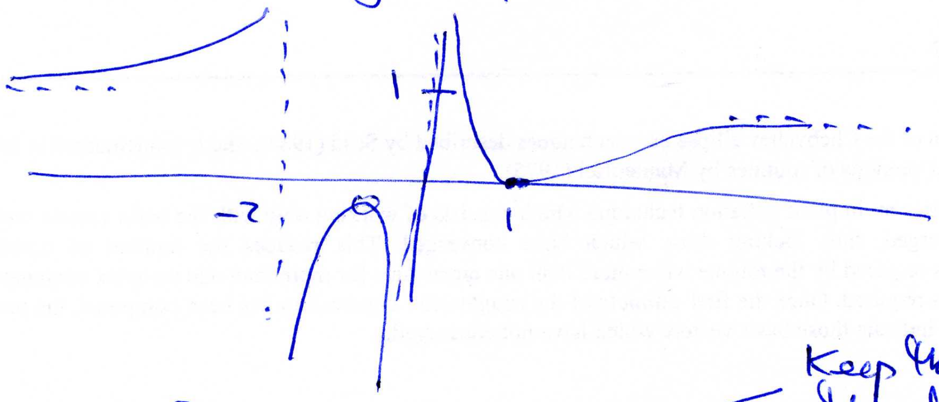
[3] (a) Double zero at $x=1$. Poles at $x=0,-2$.

As $x \to \pm\infty$, $y \to 1$.



(b) $y = \dfrac{x^2-2x+1}{x^2+2x} = \dfrac{(x-1)^2}{x^2+2x}$ ← Keep this written in this fashion.

$$\therefore \frac{dy}{dx} = \frac{(x^2+2x)\cdot 2(x-1) - (x-1)^2(2x+2)}{(\cdots)^2}$$

$$= \frac{2(x-1)\left[(x^2+2x)-(x^2-1)\right]}{(\ )^2} = \frac{2(x-1)(2x+1)}{(\ )^2}$$

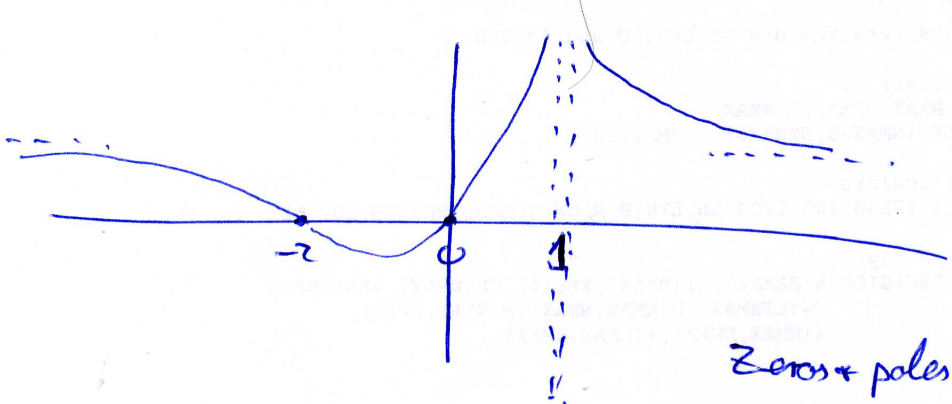
$\therefore$ Critical points at $x=1$ and $x=-\frac{1}{2}$

(c) $$I = \int_1^2 \frac{x^2-2x+1}{x^2+2x}\,dx = \int_1^2 \frac{(x^2+2x)+(1-4x)}{x^2+2x}\,dx$$

$$= \int_1^2 \left[1 + \frac{1}{2x} - \frac{9/2}{x+2}\right]dx = \left[x + \tfrac{1}{2}\ln|x| - \tfrac{9}{2}\ln|x+2|\right]_1^2$$

$$= 1 + \tfrac{1}{2}\ln 2 - \tfrac{9}{2}\ln\tfrac{4}{3} = 0.05200 \quad (5\text{DP})$$

(d)



Zeros + poles swapped.

Large $|x|$ unchanged.

4 (a) $y = \sin x$

Taylor about $x=0$: $y = y(0) + x\,y'(0) + \dfrac{x^2}{2!}y''(0) + \dfrac{x^3}{3!}y'''(0) + \dots$

n	$y^{(n)}(x)$	$y^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1
6	$-\sin x$	0

So $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} \dots$

(b) $\operatorname{sinc} x = \left[x - x^3/3! + \dots\right]/x = 1 - x^2/3! + x^4/5! \dots$

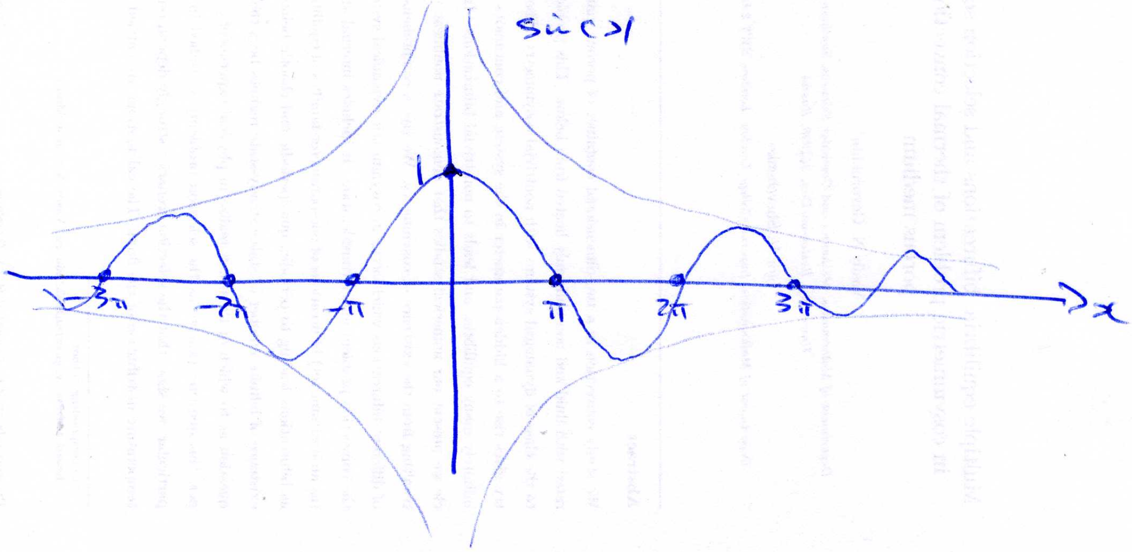
Hence $\lim_{x\to 0} \operatorname{sinc} x = 1$.

(c) $\lim_{x\to 0} \dfrac{\sin x}{x} = \lim_{x\to 0} \dfrac{\cos x}{1} = 1$ l'Hôpital.

(d) Zeros at $x = \pm\pi, \pm 2\pi, \pm 3\pi$ etc.

$x = 0 \Rightarrow \operatorname{sinc} x = 1$ from parts (b) & (c)

Envelope bounded by $\pm 1/x$



(e) $\operatorname{sinc} x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \dots$

$$\Rightarrow \int_0^X \operatorname{sinc} x \, dx = X - \frac{X^3}{3 \times 3!} + \frac{X^5}{5 \times 5!} - \frac{X^7}{7 \times 7!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n+1) \times (2n+1)!}$$

Radius of convergence: $u_n = \frac{(-1)^n X^{2n+1}}{(2n+1)(2n+1)!}$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} X^{2n+3}}{(2n+3)(2n+3)!} \, \frac{(2n+1)(2n+1)!}{(-1)^n X^{2n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{X^2}{(2n+3)(2n+1)} \times \frac{(2n+1)}{(2n+3)} \right| = 0. \ (<1)$$

True for all values of X $\Rightarrow$ infinite radius of convergence.

(f) If $y = \frac{\sin x}{x}$ then $y' = \frac{x \cos x - \sin x}{x^2}$.

So $y' = 0$ when $x \cos x = \sin x \Rightarrow \tan x = x$.

(g) Using y' in (f) we have

$$y'' = [x^2][\cancel{\cos x} - x \sin x - \cancel{\cos x}] - 2x[x \cos x - \sin x]$$

$$= \frac{(2x - x^3)\sin x - 2x^2 \cos x}{x^4}.$$

Hence $xy'' + 2y' + xy = \left[\frac{(\cancel{2x} - x^3)\sin x - \cancel{2x^2 \cos x}}{x^3} \right] + 2\left[\frac{\cancel{x^2 \cos x} - \cancel{x \cos x}}{x^3} \right] + \cancel{\sin x}$

$= 0$.