2 Complex Numbers

2.1 Historical Note [For interest only]

As mathematics has developed from early times it has suffered many revolutions in terms of the ideas which are taken as acceptable. Before and up to the time of Pythagoras (approximately 569 BC) it was believed that all quantities could be expressed in terms of whole numbers. What we now call rational fractions were allowed, since they are ratios of whole numbers (e.g. $\frac{3}{4}$), but concepts such as π and $\sqrt{2}$ were unacceptable. Pythagoras himself believed all of this, but this belief sat uncomfortably beside the fact that the diagonal of a square cannot be written in terms of a rational fraction times the side of the square. The technical term for this is that the diagonal is **incommensurate** with its side. Note that the hypotenuse of the right-angled triangle with sides, 3, 4 and 5, is commensurate with the other two sides because the ratios of those sides to the hypotenuse are $\frac{3}{5}$ and $\frac{4}{5}$, i.e. ratios of whole numbers. Neither π nor $\sqrt{2}$ can be written like this.

Returning to the diagonal of a square, Pythagoras's theorem applied to two sides of length 1 which subtend a right angle shows that the length of the hypotenuse is $\sqrt{2}$ — this is well-known. If we wish to prove that this number cannot be expressed as a ratio of whole numbers we may first assume the opposite, namely that $\sqrt{2} = n/m$ where n and m are whole numbers that do not have any common factors. If we square both sides and rearrange slightly we get

$$n^2 = 2m^2. \tag{1}$$

As the right hand side is even, this implies that n^2 is even, and, given that only even numbers have squares which are even, this implies that n must be even and so we can write $n = 2p$ where p is a whole number. On substitution into (1) we get

$$4p^2 = 2m^2 \qquad \Rightarrow \qquad m^2 = 2p^2. \tag{2}$$

In turn this latest expression implies that m is also even. But we have now shown that both n and m have a common factor, 2, which is in contradiction to the original assumption. Therefore $\sqrt{2}$ cannot be expressed as a rational fraction. Consequently it is an example of an **irrational** number.

The scientific community, Pythagoras notwithstanding, eventually accepted irrational numbers since they often arise when solving polynomial equations such as quadratic equations. It has also been shown that numbers like π and e, the base of the natural logarithm, are also irrational, but since they do not arise from solving polynomial equations they are also termed **transcendental**. Although equations such as

$$x^2 - 6x + 8 = 0 \qquad \text{and} \qquad y^2 - 6y + 7 = 0$$

may be written in the forms

$$(x-3)^2 = 1 \qquad \text{and} \qquad (y-3)^2 = 2$$

and hence have the solutions,

$$x = 2, 4 \qquad \text{and} \qquad y = 3 \pm \sqrt{2},$$

equations such as,

$$z^2 - 6z + 10 = 0, \qquad \text{or} \qquad (z-3)^2 = -1, \tag{3}$$

were deemed for a long time not to have a solution. The reason simply was that $z = 3 \pm \sqrt{-1}$ was considered to be nonsensical because there is no number whose square could possibly be equal to -1.

Such a state of affairs lasted for quite some time, and it was only around the time when Cardano (1501-1576) discovered how to solve a cubic equation of the form, $ax^3 + bx^2 + cx + d = 0$, that the resistance to $\sqrt{-1}$ finally faded. The reason for this is that, while a cubic equation may have either one real root or three real roots, Cardan's formula giving the three roots, when they are all real, involves using the square roots of negative numbers; the detailed formula is given in the next section.

The following is a quotation from *Significant Figures* by Ian Stewart which I think is of interest in this context.

> *Ars Magna* [by Cardano] is significant for one other reason. Cardano applied his algebraic methods to find two numbers whose sum is 10 and product is 40, and got the answer $5+\sqrt{-15}$ and $5-\sqrt{-15}$. Since negative numbers have no square roots, he declared this result to be 'as subtle as it is useless'. The formula for cubics also leads to such quantities when all three solutions are real, and in 1572 Rafael Bombelli observed that if you ignore what such expressions mean and just do the sums, you get the correct real solutions. Eventually this line of thinking led to the creation of the system of complex numbers in which -1 has a square root. This extension of the real number system is essential to today's mathematics, physics and engineering.

Thus it was that the symbol i (or, when within the engineering fraternity, j) came to signify the "value" of $\sqrt{-1}$, since it could at least be interpreted as a mathematical trick to obtain results which are nevertheless correct. The symbols i and j denote imaginary numbers, and, together with the real numbers, are termed complex numbers. The solution of Equation (3) for z may therefore be written in the form $z = 3 \pm j$.

It is interesting to note throughout this very brief historical sketch the type of words that have been used to denote the new kinds of numbers: irrational, transcendental and imaginary! Perhaps these mark a certain amount of distrust or perhaps awe when compared with nice safe "rational" numbers. Much more recently the term "surreal numbers" has been invented — check this out on wikipedia — and so the old tradition of curious names continues.

2.2 Cardan's formula for the cubic equation [For interest only]

The solution for the **quadratic equation** is well-known. If z satisfies

$$az^2 + bz + c = 0, \tag{4}$$

then

$$z_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad z_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}, \tag{5}$$

where, if we set $q = b^2 - 4ac$, we have the three cases

$$\begin{array}{rcc} q > 0 & \Rightarrow & \text{two real roots} \\ q = 0 & \Rightarrow & \text{two equal real roots} \\ q < 0 & \Rightarrow & \text{two complex roots.} \end{array} \tag{6}$$

The value, q, is called the discriminant: its value allows us to distinguish between the three potential cases.

The above categorisation holds only when a, b and c are real. But the formulae for z_1 and z_2 remains valid for complex values of a, b and c.

For the **cubic equation**, $z^3 + a_2 z^2 + a_1 z + a_0 = 0$, we let

$$q = \tfrac{1}{3}a_1 - \tfrac{1}{9}a_2^2 \qquad r = \tfrac{1}{6}(a_1 a_2 - 3a_0) - \tfrac{1}{27}a_2^3. \tag{7}$$

Again we have three cases:

$$\begin{array}{rcc} q^3 + r^2 > 0 & \Rightarrow & \text{one real root and a pair of complex roots} \\ q^3 + r^2 = 0 & \Rightarrow & \text{all roots are real with at least two equal} \\ q^3 + r^2 < 0 & \Rightarrow & \text{all roots are real.} \end{array} \tag{8}$$

To complete the solution we set

$$s_1 = [r + \sqrt{(q^3 + r^2)}]^{1/3} \qquad s_2 = [r - \sqrt{(q^3 + r^2)}]^{1/3} \tag{9}$$

and then

$$\begin{aligned} z_1 &= (s_1 + s_2) - \tfrac{1}{3}a_2 \\ z_2 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 + \tfrac{1}{2}\sqrt{3}(s_1 - s_2)j \\ z_3 &= -\tfrac{1}{2}(s_1 + s_2) - \tfrac{1}{3}a_2 - \tfrac{1}{2}\sqrt{3}(s_1 - s_2)j. \end{aligned} \tag{10}$$

There is a much longer formula for the solution of quartic equations (i.e. fourth order polynomials) but there are no general formulae for the solution of quintics or of higher order polynomials. This was proved in 1839 by Abel.

That completes the historical stuff! From now on we shall look at the properties of complex numbers, determine how they may be used and manipulated, and then finally we'll consider a general method for finding the roots of complex numbers (by which we mean the fractional powers of complex numbers such as square roots).

2.3 Manipulation of Complex Numbers

The basic rules for adding, subtracting, multiplying and dividing complex numbers are not too far removed from those involving solely real numbers, except that we will often need to use the result that $j^2 = -1$, and that real and imaginary numbers are usually treated separately.

Addition and Subtraction:
Here we add and subtract in the way one would expect by collecting like terms. For example

$$(1+5j)+(2-3j) \quad = \quad (1+2)+(5-3)j \quad = \quad 3+2j, \tag{11}$$

and

$$(1+5j)-(2-3j) \quad = \quad (1-2)+(5+3)j \quad = \quad -1+8j. \tag{12}$$

So these work in the same way as $(x+5y)+(2x-3y) = 3x+2y$.

Multiplication:
Here we expand the product in the same way as we do for the product,

$$(a+b)(c+d) = ac+bd+ad+bc,$$

except that a little tidying up takes place afterwards using $j^2 = -1$. In general, we have,

$$(a+bj)(c+dj) = ac+bd\,j^2+adj+bcj = (ac-bd)+(ad+bc)j. \tag{13}$$

A numerical example:

$$\begin{aligned} (1+5j)(2+3j) \quad &= (1\times 2)+(5j\times 3j)+(1\times 3j)+(5j\times 2) \\ &= 2+15\,j^2+3j+10j \\ &= 2-15+13j \\ &= -13+13j. \end{aligned} \tag{14}$$

Note: Integer powers of complex numbers (like squares and cubes and 10th powers etc.) may be evaluated this way but there is a better way which will follow later.

Division:
This is the most complicated operation of the four and it is described best using the concept of the complex conjugate. If $z = x + yj$, then the **complex conjugate** of z is written as $\overline{z}$ and defined as

$$\overline{z} = x - yj. \tag{15}$$

Clearly $\overline{3+4j} = 3 - 4j$, $\overline{6} = 6$ and $\overline{5j} = -5j$. One very useful consequence of the complex conjugate is that the product, $z\overline{z}$, is real:

$$\begin{aligned} z\overline{z} &= (x+yj)(x-yj) && \text{by definition} \\ &= x^2 - y^2 j^2 + xyj - xyj && \text{multiplying out} \\ &= x^2 + y^2 && \text{using } j^2 = -1. \end{aligned} \tag{16}$$

This provides us with a sneaky way for evaluating the quotient of two complex numbers: we multiply the denominator and the numerator by the complex conjugate of the denominator, and this guarantees that the new denominator is real. Here is an example:

$$\begin{aligned} \frac{1}{3+4j} &= \frac{1}{3+4j} \times \frac{3-4j}{3-4j} \\ &= \frac{3-4j}{(3+4j)(3-4j)} \\ &= \frac{3-4j}{25} \\ &= \frac{3}{25} - \frac{4}{25}j. \end{aligned} \tag{17}$$

In general we have

$$\frac{1}{x+yj} = \frac{x-yj}{(x+yj)(x-yj)} = \frac{x-yj}{x^2+y^2}. \tag{18}$$

Another numerical example is:

$$\frac{2+j}{1+j} = \frac{(2+j)(1-j)}{(1+j)(1-j)} = \frac{3-j}{2}.$$

Note: This idea of using a complex conjugate may also help us in an entirely different context, one which involves only real numbers. Try this one out:

$$\begin{aligned} \frac{8}{1+\sqrt{5}} &= \frac{8}{(1+\sqrt{5})} \times \frac{(1-\sqrt{5})}{(1-\sqrt{5})} \\ &= \frac{8(1-\sqrt{5})}{1-5} = 2(\sqrt{5}-1). \end{aligned} \tag{19}$$

2.4 Geometrical Interpretation

This will give us some further physical understanding of complex numbers, but will also allow us to do things like finding square roots, tenth roots and $2/5$th powers of complex numbers.

The complex number $z = a + bj$ may be represented as the coordinates of a point in what is called the **complex plane**:

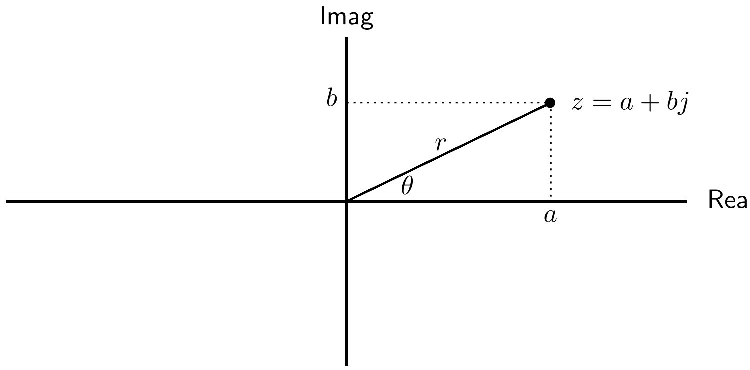


Figure 2.1. Showing the location of the complex number, $z = a + bj$, in the Argand Diagram (or Complex Plane). Also shown are $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z) = \tan^{-1}(b/a)$.

The horizontal axis (abscissa) is called the **real axis**, while the vertical axis (ordinate) is called the **imaginary axis**. The whole Figure is also referred to as the **Argand diagram**.

The length of the line joining the origin to z is $\sqrt{a^2 + b^2}$ which is precisely $\sqrt{z\overline{z}}$; this is called the **modulus** of z and is denoted by $|z|$.

The angle that the line joining the origin to z makes with the real axis (measured anti-clockwise) is called the **argument** of z, and is denoted as $\arg z$. If we let $\theta = \arg(z)$ then $\tan\theta = b/a$. Conversely we have $\arg(z) = \tan^{-1}(b/a)$, although it is necessary to specify which of the two possible angles is correct. For example, if $b = a = 1$, then the diagram indicates that $\arg z = \pi/4$, but the inverse tan formula simply says that $\theta = \tan^{-1}(1)$ for which we may have either $\theta = \pi/4$ or $\theta = 5\pi/4$; therefore we either need to state in addition that we require θ to lie in the first quadrant (i.e. $0 \le \theta \le \pi/2$), or else quote the correct value in radians. **We note that θ is always measured in the anticlockwise direction.**

The conjugate of z may be located by taking the mirror image of z in the real axis; see Figure 2.2. From this diagram we see that $|z| = |\overline{z}|$ and that $\arg(z) = -\arg(\overline{z})$.

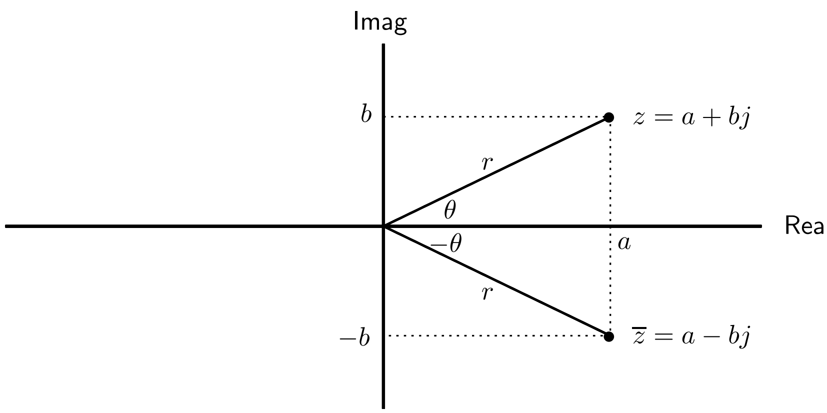


Figure 2.2. Showing the locations of the complex number, $z = a + bj$ and its conjugate, $\overline{z} = a - bj$, in the Complex Plane.

Addition: In the Argand diagram this appears to follow the same rule as the addition of two vectors in that the sum is obtained by completing the rhombus; see Figure 2.3.

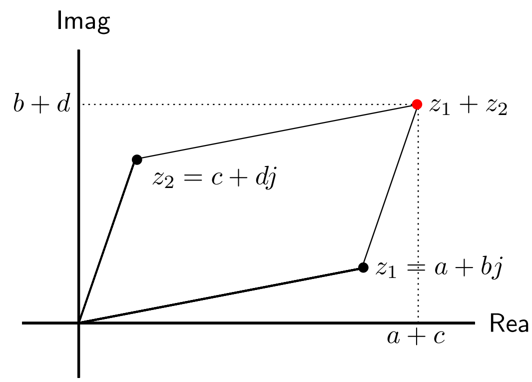


Figure 2.3. Showing how two complex numbers, $z_1 = a+bj$ and $z_2 = c+dj$, may be added by completing the rhombus. Equivalently, we add the real and imaginary parts separately.

Multiplication: For example, we have $(2+4j)\times(1+j) = -2+6j$. On the Complex Plane we have:

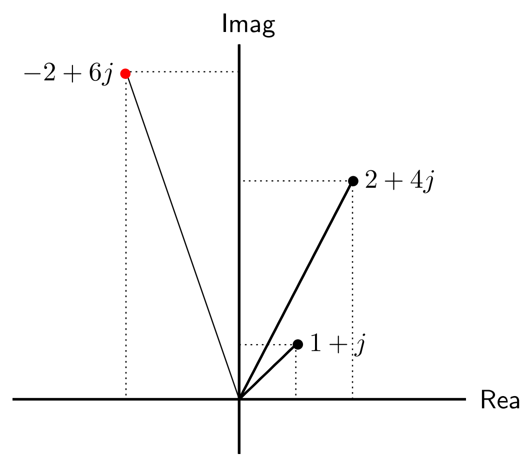


Figure 2.4. Showing the multiplication of the two given complex numbers.

From this Figure it is not immediately obvious what is happening geometrically when we multiply complex numbers. However, if we find the moduli of each of the three numbers:

$$|2+4j| = \sqrt{20}, \qquad |1+j| = \sqrt{2}, \qquad |-2+6j| = \sqrt{40} = \sqrt{2}\times\sqrt{20},$$

then we see that the modulus of the product is equal to the product of the moduli. It is also clear from the Argand diagram that the argument of the product is greater than the (positive) argument of individual numbers forming the product.

We may investigate this in more detail by first considering a partial translation of a complex number into a polar coordinate form. In Fig. 2.1 we can see that

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta,$$

or that,

$$a+bj = r(\cos\theta + j\sin\theta). \tag{20}$$

Therefore we may write down similar general forms for the following two complex numbers:

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + j\sin\theta_2). \tag{21}$$

Clearly, the moduli of these numbers are r_1 and r_2, respectively, and their arguments are θ_1 and θ_2. Hence

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2), \\ &= r_1 r_2 \Big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\Big], \\ &= r_1 r_2 \Big[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)\Big]. \end{aligned} \tag{22}$$

where the last line was formed by appealing to some multiple-angle formulae.

This result shows that the **modulus of the product is equal to the** ***product*** **of the moduli**, but also that the **argument of the product is equal to the** ***sum*** **of the arguments**. This leads us into what is called the polar (or exponential or complex exponential) representation of complex numbers.

2.5 Complex Exponential form

Recalling the expression given in Eq. (20), we may write down the following:

$$\begin{aligned} z &= a + bj && \text{by definition} \\ &= (r\cos\theta) + j(r\sin\theta) && \text{using Eq. (20)} \\ &= r(\cos\theta + j\sin\theta) \\ &= re^{j\theta}. \end{aligned} \tag{23}$$

This last step of rewriting the coefficient of r as a complex exponential needs a little explanation, but this cannot be done satisfactorily without resorting to either Taylor Series, which is a device for writing functions as an infinitely long polynomial, or to the theory of Ordinary Differential Equations; these will be covered later in ME10304 this semester and ME10305 (Mathematics 2) next semester, respectively. You will need to take Eq. (23) on trust for now. . . .

Thus complex numbers may be written in either Cartesian form $z = a + bj$ or Polar/Exponential form $z = re^{j\theta}$. [The word, Cartesian, is always spelt with a capital because it is named for Descartes, otherwise known as Cartesius in its Latinized form.] An alternative notation is $z = r\angle\theta$; this is often used in Electrical Engineering contexts, where θ can be allowed to be in degrees. Another occasional notation is $r\,\text{cis}\,\theta$, where cis is a shorthand for $\cos\theta + i\sin\theta$ — this was invented by Hamilton in 1866, but is rarely used now. So the complex exponential form is the generally accepted way.

Warning: When writing $re^{j\theta}$, the angle, θ, **must** always be measured in radians. The same is true for sine and consine. The reason is that derivatives are incorrect when using degrees.

Addition and subtraction are most easily carried out in Cartesian form, while multiplication and division are most easily carried out in Polar form:

$$z_1 z_2 = \Big[r_1 e^{j\theta_1}\Big]\Big[r_2 e^{j\theta_2}\Big] = (r_1 r_2)\, e^{j(\theta_1+\theta_2)},$$

$$z_1/z_2 = \Big[r_1 e^{j\theta_1}\Big]\Big/\Big[r_2 e^{j\theta_2}\Big] = (r_1/r_2) e^{j(\theta_1-\theta_2)}.$$

That said, multiplication and division while in Cartesian form are also pretty straightforward.

The conversion of a complex number from polar/exponential form to Cartesian form is straightforward if one remembers the geometry of the Argand diagram:

$$2e^{j\pi/4} = 2\Big[\cos(\pi/4) + j\sin(\pi/4)\Big] = 2\Big(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\Big) = \sqrt{2} + \sqrt{2}j.$$

But the conversion from Cartesian to polar/exponential form requires a little more care due to the fact that the inverse tangent function has more than one solution. For example, $\tan^{-1} 1 = \frac{1}{4}\pi$ or $\frac{5}{4}\pi$. Thus conversion from Cartesian to polar form also requires one to choose the correct inverse tangent.

Example 2.1: Express $-1 + 2j$ in complex exponential form.

We have $-1+2j = re^{j\theta}$. Here $r^2 = (-1)^2 + (2)^2 = 5 \;\Rightarrow\; r = \sqrt{5}$. The argument satisfies $\tan\theta = (2)/(-1) = -2$, and therefore θ is either -1.107149 (directly from my calculator) or 2.034444 (which is π plus the first value). Given where the complex number lies in the complex plane, i.e. the second quadrant, then $\theta = 2.034444$ or 116.565°.

Note: It is usual to quote angles to lie in the range, $-\pi < \theta \le \pi$ although there may very occasionally be the need to use $0 \le \theta < 2\pi$.

2.6 de Moivre's Theorem

This is a very useful result and is known as a theorem despite the fact its proof takes only two lines (i.e. just one step). It is based on the following well-known property of the powers of exponentials,

$$(a^b)^c = a^{bc}. \tag{24}$$

We apply this result to $e^{\theta j}$:

$$\begin{aligned} (e^{\theta j})^n &= e^{n\theta j} \\ \Rightarrow \qquad \Big[\cos\theta + j\sin\theta\Big]^n &= \cos(n\theta) + j\sin(n\theta). \end{aligned} \tag{25}$$

From this we may recover the various multiple angle formulae from trigonometry. For example, when $n = 2$ we have

$$\begin{aligned} \cos 2\theta + j\sin 2\theta &= \Big[\cos\theta + j\sin\theta\Big]^2 \\ &= [\cos^2\theta - \sin^2\theta] + j[2\sin\theta\cos\theta]. \end{aligned} \tag{26}$$

On equating real and imaginary parts we get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\sin\theta\cos\theta.$$

When $n = 3$, we have

$$\begin{aligned} \cos 3\theta + j\sin 3\theta &= \Big[\cos\theta + j\sin\theta\Big]^3 \\ &= \cos^3\theta + 3j\cos^2\theta\sin\theta + 3j^2\cos\theta\sin^2\theta + j^3\sin^3\theta \\ &= [\cos^3\theta - 3\cos\theta\sin^2\theta] + j[3\cos^2\theta\sin\theta - \sin^3\theta]. \end{aligned} \tag{27}$$

This final line used the fact that $j^3 = j^2 j = -j$, and it may also be tidied up to give

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \qquad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

2.7 Roots of complex numbers

It is well-known that there are two square roots of positive numbers and, when that number is 9, then those roots are 3 and -3. But both negative numbers and complex numbers also have two square roots. So let us check out a few examples of square roots and other roots.

Example 2.2. Given that both $j^2 = -1$ and $(-j)^2 = -1$, it is clear that the two square roots of -1 are $\pm j$.

Example 2.3. Likwise, given that $(1+j)^2 = 2j$ and $(-1-j)^2 = 2j$, then the square roots of $2j$ are $\pm(1+j)$.

Example 2.4. Once more, given that $(2+j)^2 = 3+4j$ and $(-2-j)^2 = 3+4j$, then the square roots of $3+4j$ are $\pm(2+j)$.

The pattern so far is that each square root is precisely the negative of the other, which is exactly the same as for the square roots of positive numbers. No surprise there! Here's a different case:

Example 2.5. We have,

$$(1+j)^4 = -4, \qquad (-1-j)^4 = -4, \qquad (1-j)^4 = -4 \qquad \text{and} \qquad (-1+j)^4 = -4.$$

Therefore we have shown that -4 has four fourth roots:

$$(-4)^{1/4} = \pm 1 \pm j,$$

where all four possible combinations of plus/minus may be taken.

Example 2.6. One more case with a fourth power:

$$(1+2j)^4 = -7-24j, \qquad (-1-2j)^4 = -7-24j,$$

$$(2-j)^4 = -7-24j \qquad \text{and} \qquad (-2+j)^4 = -7-24j.$$

Therefore the four fourth roots of $(-7-24j)$ are

$$\pm(1+2j) \qquad \text{and} \qquad \pm\,(2-j).$$

Perhaps this still isn't sufficient to see the full pattern yet, although the fact that complex numbers have two square roots and four fourth roots does suggest something. So here is something a little different:

Example 2.7. Consider the following:

$$(-2)^3 = -8, \qquad (1+\sqrt{3}j)^3 = -8 \qquad \text{and} \qquad (1-\sqrt{3}j)^3 = -8.$$

So we have three third roots of -8, namely

$$-2, \quad 1+\sqrt{3}j \quad \text{and} \quad 1-\sqrt{3}j$$

.

Example 2.8. If one were to find the sixth powers of 2, -2, $1+\sqrt{3}j$, $1-\sqrt{3}j$, $-1+\sqrt{3}j$ and $-1-\sqrt{3}j$, then each would yield 64. So 64 has six sixth roots.

So that is a pattern, namely that there will be n n^{th} roots for a complex number. However, that is not the only pattern. The following Figure shows how the roots given within each example above are related to one another.

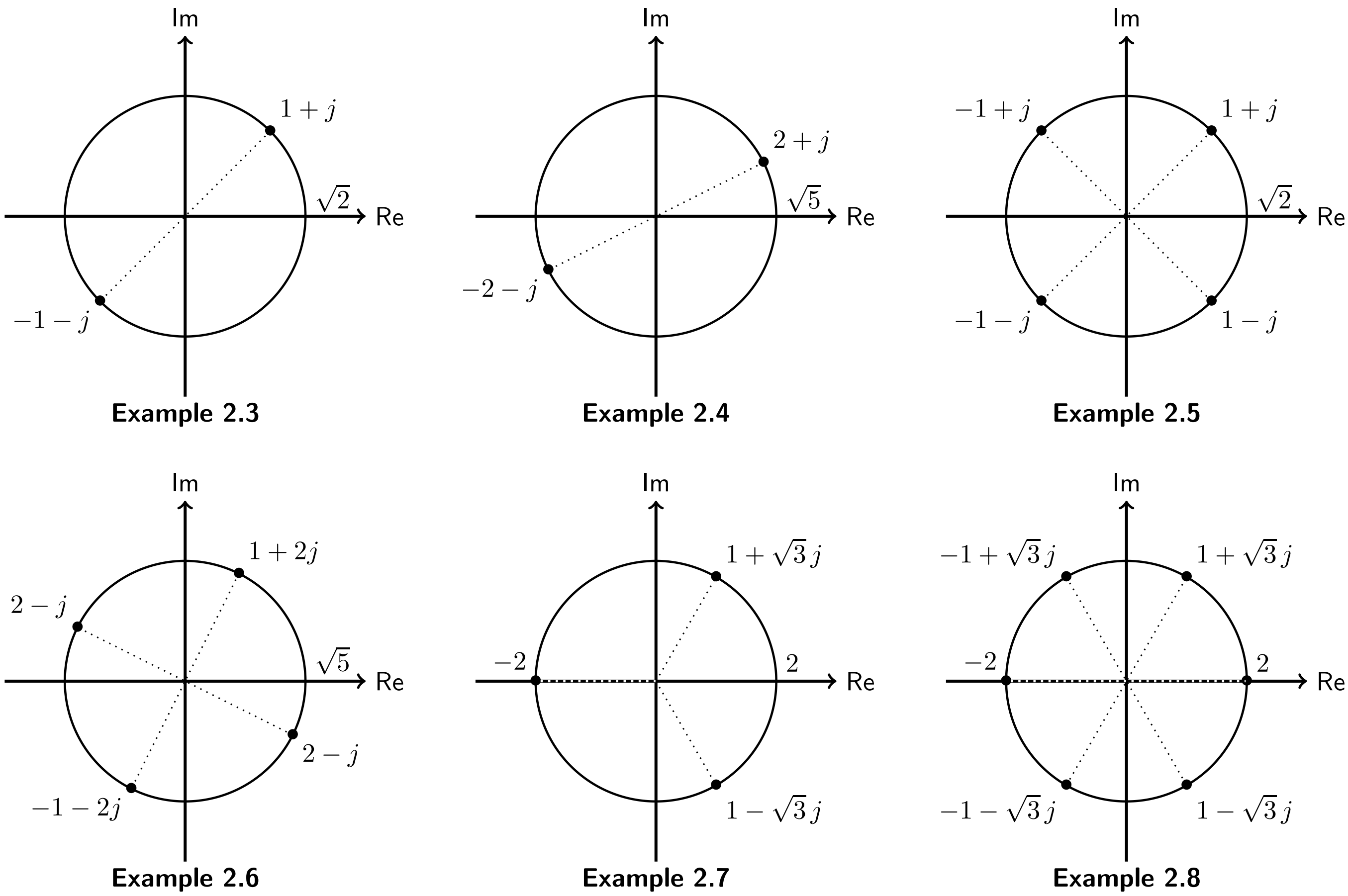


Figure 2.5. Displaying the roots corresponding to Examples 2.3 to 2.8.

The pattern that we need may now be seen in Fig. 2.5 and, for each case, the circle on which all the roots lie is split into equal segments.

The general method. We may motivate the general method using the data in Example 2.5. The four roots are given by

$$\sqrt{2}e^{j\pi/4}, \quad \sqrt{2}e^{3j\pi/4}, \quad \sqrt{2}e^{5j\pi/4} \quad \text{and} \quad \sqrt{2}e^{7j\pi/4}. \tag{28}$$

If we raise all of these to the 4th power, then we obtain,

$$4e^{j\pi}, \quad 4e^{3j\pi}, \quad 4e^{5j\pi} \quad \text{and} \quad 4e^{7j\pi}, \tag{29}$$

respectively. The arguments of these numbers are π, 3π, 5π and 7π, i.e. they are separated by 2π, and hence each one represents the same point in the Complex Plane. We may write these four values in the alternative form:

$$4e^{(1+2n)\pi j}, \qquad n = 0, 1, 2, 3, \tag{30}$$

and therefore the roots listed in Eq. (28) may be written as,

$$z_n = \sqrt{2}e^{(1+2n)\pi j/4}, \qquad n = 0, 1, 2, 3. \tag{31}$$

Note: In Eq. (31) n takes four successive values. Given the way we have motivated the methd it is natural that the first value of n will be zero. However, any four successive values of n will be ok.

Note: The general method for finding roots uses the above approach but does it backwards! Three examples follow.

Example 2.9. Find the square roots of j.

We may write j in polar form as either $e^{(\pi/2)j}$ (the obvious one where the argument is $\pi/2$) or as $e^{(5\pi/2)j}$. More generally it is $e^{(\pi/2+2n\pi)j}$ where $n=0$ or $n=1$. These two forms for j lie at the same point in the Complex Plane and must do so! However, they will cease to do so when we take the square roots:

$$\sqrt{j} \;=\; e^{(\pi/4)j} \qquad \text{or} \qquad e^{(5\pi/4)j}. \tag{32}$$

For this particular example we can write the Cartesian form without the use of a calculator! We have,

$$\begin{aligned}\sqrt{j} &= \Big(\cos\tfrac{1}{4}\pi + j\sin\tfrac{1}{4}\pi\Big) \qquad \text{or} \qquad \Big(\cos\tfrac{5}{4}\pi + j\sin\tfrac{5}{4}\pi\Big) \\ &= \pm\Big(\frac{1+j}{\sqrt{2}}\Big).\end{aligned} \tag{33}$$

Example 2.10. Find the fifth roots of $(3+4j)$.

In polar form we have

$$z = 5e^{\theta j},\ 5e^{(\theta+2\pi)j},\ 5e^{(\theta+4\pi)j},\ 5e^{(\theta+6\pi)j},\ 5e^{(\theta+8\pi)j}$$

where we have written five different expressions for z, and where $\tan\theta = \frac{4}{3}$ with θ lying in the first quadrant (i.e. $\theta = 0.927295$). The five fifth roots are

$$z = 5^{1/5}e^{(\theta+2n\pi)j/5} \quad \text{for } n = 0,1,2,3,4.$$

If we were to plot these in the Complex Plane then we would see that the five points are equally distributed around a circle of radius $5^{1/5}$ centred on the origin.

Example 2.11. Find $z = (4+3j)^{2/3}$.

For such a power, it is probably better to find the square first and then take the cube root, although it could be done the other way around. Perhaps I ought to show that they give the same solutions. So we'll square first...

First way: Let

$$z = \Big[(4+3j)^2\Big]^{1/3} = (7+24j)^{1/3}.$$

In polar form we have $7+24j = 25e^{(\theta+2n\pi)j}$ for $n=0,1,2$, where $\tan\theta = \frac{24}{7}$ and θ lies in the first quadrant, we note that the modulus of $(7+24j)$ is 25. Therefore

$$z = 25^{1/3}e^{(\theta+2n\pi)j/3} \quad \text{for } n = 0,1,2 \text{ and where } \theta = 1.287002. \tag{34}$$

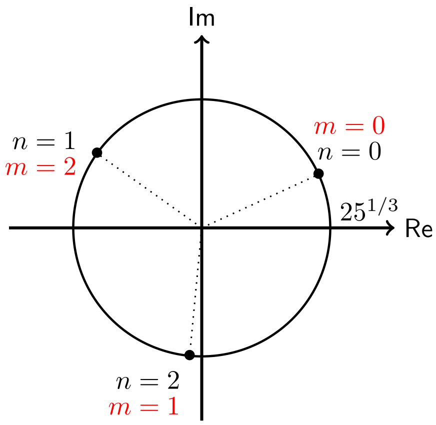
Second way: Let

$$z = (4+3j)^{2/3}.$$

In polar form we have $4+3j = 5e^{(\phi+2m\pi)j}$ for $m = 0, 1, 2$, where $\tan\phi = \frac{3}{4}$ and where ϕ also lies in the first quadrant. We have used m here instead of n, and ϕ instead of θ to distinguish the two analyses. The modulus of $(4+3j)$ is 5. Therefore

$$z = 5^{2/3}e^{2(\phi+2m\pi)j/3} \quad \text{for } m = 0, 1, 2 \text{ and where } \phi = 0.643501. \tag{35}$$

The following figure shows where the roots are as a consequence of these two analyses. The locations of the roots are identical, but manner in which they are counted are different.



Example 2.11

Note: Again I emphasize in the strongest terms that θ MUST be given in radians when using polar form.

2.8 Irrational exponents [For interest only]

Here we are thinking of powers such as $z^{\sqrt{2}}$.

We note that irrational numbers, such as $\sqrt{2}$, may not be expressed as the ratio of two whole numbers, whereas the method we have used above depends on this. Although $\sqrt{2}$ may be approximated by a rational fraction as closely as we might desire, then the more accurate the approximation is the larger is the denominator. For example, the absolute error in the fraction, $\frac{99}{70} = 1.41428571$ (8DP), is less than 10^{-4} and it is the first fraction (as the denominator increases) for which the error is smaller than 10^{-4}. For the errors, 10^{-6} and 10^{-8}, the corresponding fractions are,

$$\sqrt{2} \simeq \tfrac{1393}{985} = 1.41421320 \text{ (8DP)} \qquad \text{and} \qquad \sqrt{2} \simeq \tfrac{19601}{13860} = 1.41421356 \text{ (8DP)}.$$

Therefore increasingly accurate representations of $\sqrt{2}$ result in an increasing number of legitimate roots on a circle in the complex plane: 70, 985 and 13860 in our three examples. But typically the context will dictate if this is sensible, and it generally will not be. Therefore, in such situations we may restrict ourselves to what is called the **principal value** of the complex exponential. By this is meant that, if $z = re^{\theta j}$, then $z^{\sqrt{2}} = r^{\sqrt{2}}e^{\sqrt{2}\theta j}$. However, do be assured that such matters will not arise in engineering applications.

2.9 Relationship with the hyperbolic functions.

The functions $\cos\theta$ and $\sin\theta$ are known as either trigonometric or circular functions, while $\cosh\theta$ and $\sinh\theta$ are the hyperbolic functions. Given that

$$e^{\theta j} = \cos\theta + j\sin\theta \qquad \text{and} \qquad e^{-\theta j} = \cos\theta - j\sin\theta,$$

we may add and subtract these to obtain the relations,

$$\cos\theta = \tfrac{1}{2}\Big[e^{\theta j} + e^{-\theta j}\Big] \qquad \text{and} \qquad \sin\theta = \tfrac{1}{2j}\Big[e^{\theta j} - e^{-\theta j}\Big].$$

These are reminiscent of

$$\cosh\theta = \tfrac{1}{2}\Big[e^{\theta} + e^{-\theta}\Big] \qquad \text{and} \qquad \sinh\theta = \tfrac{1}{2}\Big[e^{\theta} - e^{-\theta}\Big],$$

and explains why there is so much similarity between the circular and hyperbolic functions, particularly when dealing with calculus and differential equations, even though the circular and the hyperbolic functions look so different from one another.