1 Curve sketching

1.1 Preamble

The chief aim of this short section of the Maths 1 unit is to develop one's understanding of what shapes the various standard functions have so that more complicated functions may be sketched. We will attempt to sketch functions by looking at the detailed formula for that function. It is hoped that this will allow one to be able to give a reasonable guess for what function represents a given graph, the reverse process. We'll begin with polynomials and then complicate issues with the modulus function, exponentials, envelopes, square roots, and finally ratios of polynomials.

We will not employ any methods from calculus in this section. This is simply because we wish to determine the rough outline of what the graph looks like, but not precisely where maxima or minima are located. We will often need to ask questions such as, where are the zeroes, where are the asymptotes, what is the large-x behaviour?

Graphs have been sketched by hand and scanned into this document — this is not because I am lazy, but rather it indicates how I require you to do it, particularly in the exam! The sketches will show all the salient features of the graph.

1.2 Polynomials

By polynomials I mean seemingly straightforward functions of x, say, such as those given by $y = x^3 - x$ and $y = (x-5)^2(x-1)x$. We will start from the utterly trivial and make our way to slightly more complicated shapes.

The straight line, $y = x$, should be quite straightforward! The bullet indicates the single zero or root.

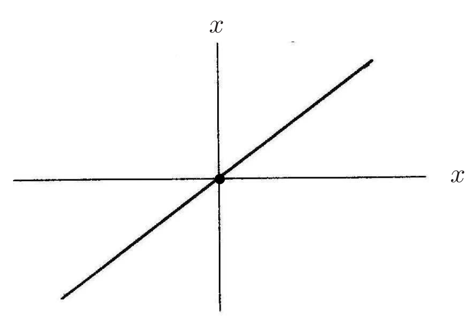

Figure 1.1.

This is a parabola, of course. It has a zero slope at $x = 0$. From the point of view of factorization, $y = x \times x$, we may say that it also has a 'double zero' at $x = 0$. This will be important below. The bottom of the parabola (double bullet) shows what a double zero looks like.

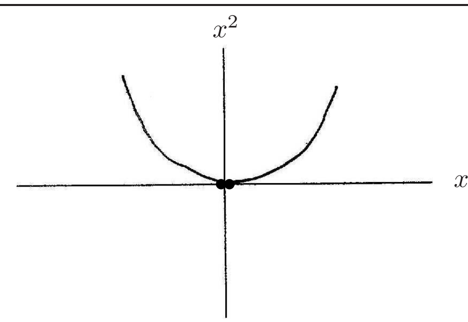

Figure 1.2.

This is a downward-facing parabola. All parabolae, $y = ax^2 + bx + c$, will look like this when $a < 0$, although the maximum will not be at the origin in general.

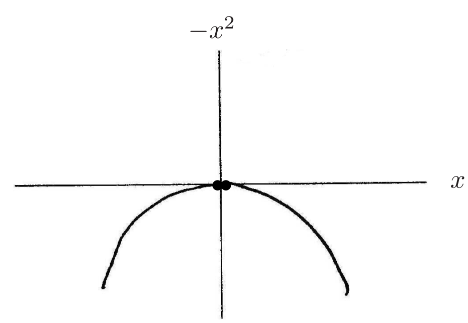

Figure 1.3.

This parabola is identical to the one in Figure 1.2, but has been shifted by 1 in the x-direction. We see that there is a double root at $x = 1$ — note the repeated $(x-1)$ factor.

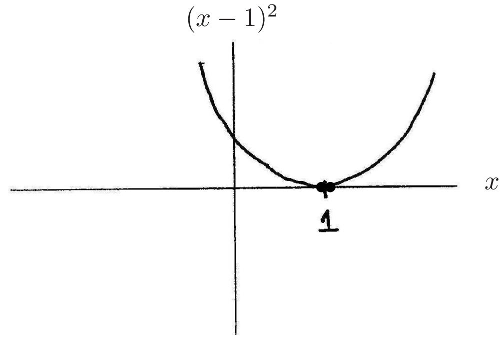

Figure 1.4.

This parabola is identical to the one in Figure 1.4, but it has been shifted downwards by 1. The resulting expression for y may be factorised into $y = x(x-2)$ and hence there are two zeros: $x = 0$ and $x = 2$.

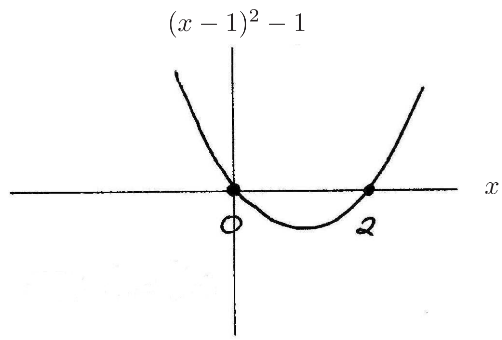

Figure 1.5.

Cubic functions

We now move on to cubic functions. These are of the general form, $y = ax^3 + bx^2 + cx + d$ where a must be nonzero, otherwise the function is a quadratic. The general shape that a cubic displays comes in three varieties which will be illustrated in the next three Figures. They are: (i) with a local maximum and a local minimum; (ii) with a point of inflexion; (iii) without a maximum, minimum or point of inflexion.

This cubic has a local maximum and a local minimum. Given that it may be factorised into the form, $y = x(x-1)(x+1)$, it has the three roots, $x = -1, 0, 1$. Note that cubics of this type don't always have three roots: if this cubic were moved upwards by 100 to give, $y = x^3 - x + 100$, then there would only be one root.

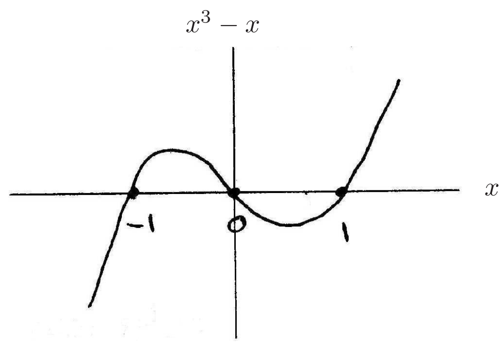

Figure 1.6.

This is the standard cubic function. It has a point of inflexion at $x = 0$, which means that both the slope and the second derivative are zero there. Given that we may write this as $y = x \times x \times x$, we see that a point of inflexion sitting on the horizontal axis corresponds to a triple zero.

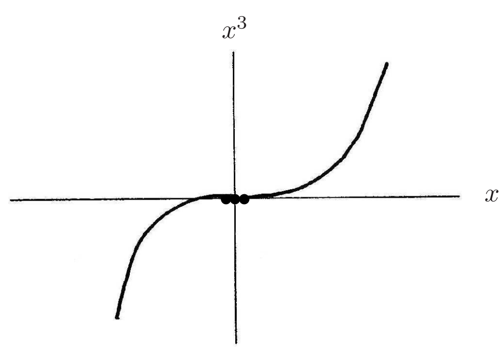

Figure 1.7.

Cubics such as this have no critical points, by which I mean maxima, minima or points of inflexion. They always have one zero, though. In this case it is at $x = 0$.

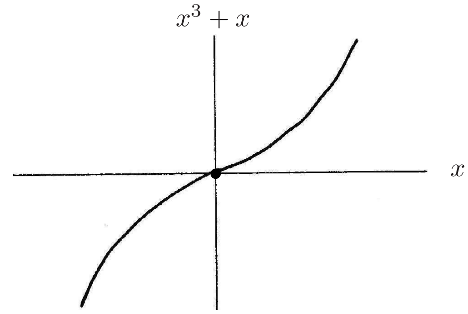

Figure 1.8.

Quartic functions

Last of all we will deal with quartic functions — note the difference between the words, *quadratic* and *quartic*. These have the general form, $y = ax^4 + bx^3 + cx^2 + dx + e$ where a is again nonzero. Given the number of constants, we could also write,

$$y = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = \sum_{n=0}^{4} a_nx^n.$$

As the largest exponent increases, we obtain more options for the types of curve, and therefore we will stop at quartics. Quartics come in the following varieties: (i) two maxima and one minimum (or vice versa); (ii) a point of inflexion and either a maximum or a minimum; (iii) a quartic minimum (or maximum); (iv) a standard parabolic minimum or maximum.

This quartic curve has three extrema: two minima and one maximum. It also has four zeros, namely $x = -2,\ -1,\ 1$ and 2, which may be found directly from the function itself. Note that if we were to add exactly the right constant (it turns out to be $9/4$) to this function in order to raise the minima so that they both lie on the x-axis, then we would now have a pair of double zeros.

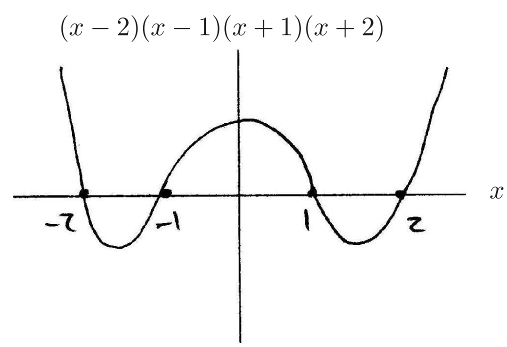

Figure 1.9.

This curve has four zeros: $x = 0,\ 0,\ 0$ and 2. Therefore there is a point of inflexion at $x = 0$ (i.e. a triple zero) and a simple zero at $x = 2$.

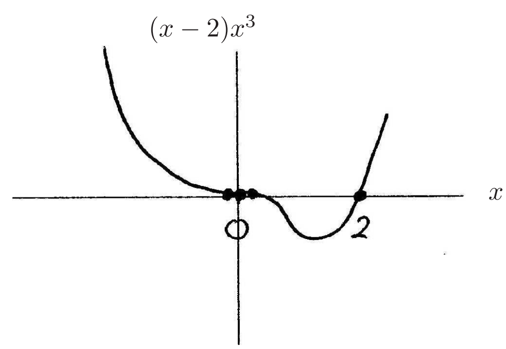

Figure 1.10.

The pure quartic function: $y = x^4$. The first three derivatives are zero at $x = 0$ and therefore this curve has a much flatter base than the parabola does, and this must be shown clearly in the sketch. We also have a four-times repeated root, and therefore $x = 0$ is a quadruple zero.

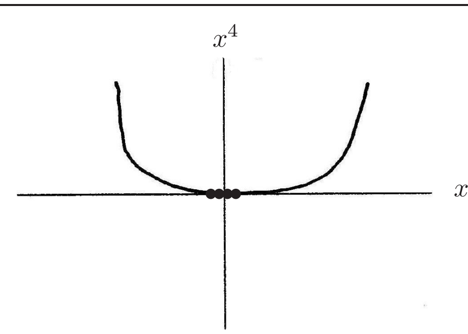

Figure 1.11.

This looks like a parabola because the x^2 term is much larger than x^4 is when x is small. The value, $x = 0$, corresponds to a double zero because $x^4 + x^2 = x^2(x^2 + 1)$. However, the function grows much faster as x increases than a parabola does because of the x^4 term, although this is quite difficult to show on a simple sketch.

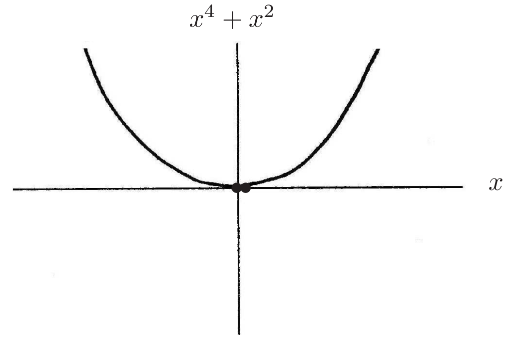

Figure 1.12.

This quartic curve is very similar to that displayed in Figure 1.9. While the general shape is the same, this one has single zeros at $x = \pm 1$ and a double zero at $x = 0$ because $x^4 - x^2 = x^2(x + 1)(x - 1)$.

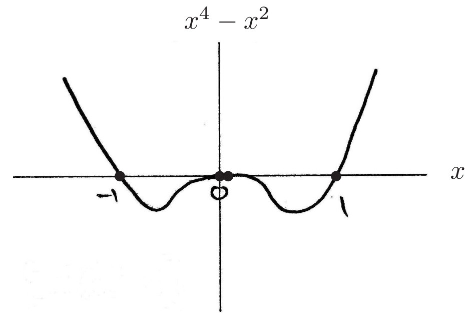

Figure 1.13.

This final quartic curve has been inferred from the given factorisation. There is a double zero at $x = -2$ and single zeros at $x = -1$ and $x = 1$. Given that the coefficient of x^4 is positive, the function becomes large and positive when $x \to \infty$. The sketch of $-(x^2 - 1)(x + 2)^2$ is the present one turned upside-down.

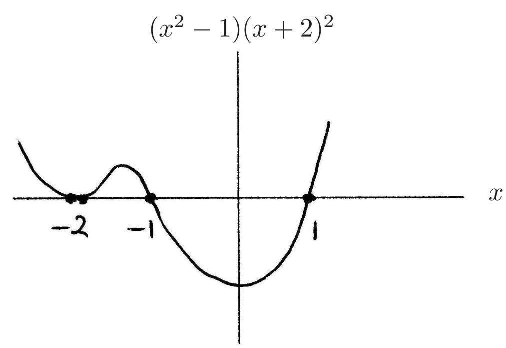

Figure 1.14.

1.3 Moduli

The modulus of a function is its absolute value. We may write this mathematically as follows:

$$|f(x)| = f(x) \quad \text{when} \quad f(x) \geq 0, \qquad |f(x)| = -f(x) \quad \text{when} \quad f(x) \leq 0.$$

We could also define it as the positive square root of the square of the function:

$$|f(x)| = +\sqrt{[f(x)]^2}.$$

Therefore the modulus function is always either positive or zero. Colloquially we speak of $|f(x)|$ as 'mod f' or 'mod f of x'. A few examples follow.

We see that $|x|$ is positive everywhere except at the origin where it is zero.

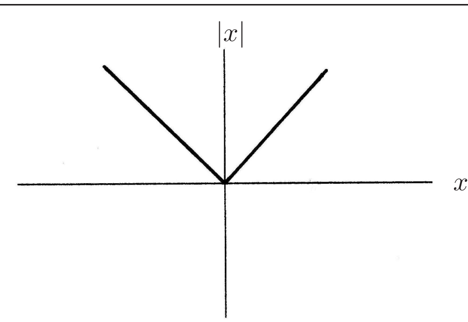

Figure 1.15.

This is 'modulus' version of Figure 1.9. I have constructed this by drawing the curve given in Figure 1.9 using short dashes. Those values which are positive have been overdrawn with a continuous line, while those parts which are negative have been multiplied by -1 and then they are drawn in with a continuous line.

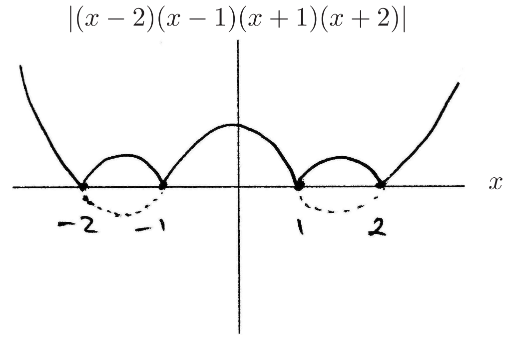

Figure 1.16.

This has been constructed in the same way as for the previous Figure. For this particular function, we may also refer to it as a 'rectified sine wave' — this is often used in Electrical Engineering.

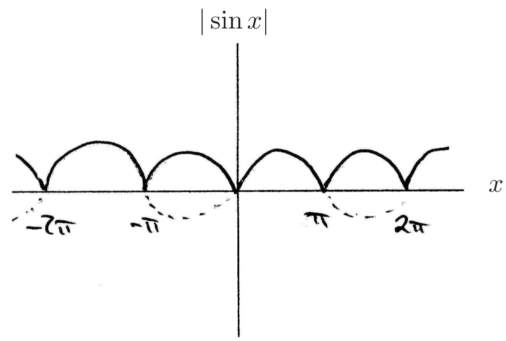

Figure 1.17.

1.4 Exponential and hyperbolic functions.

The exponential function is typified by e^x where $e = 2.7182818284590452$ to 16 decimal places. This strange number arises in many places. One of the most common is the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

a result which we will prove later in the unit, but if we set $x = 1$ we obtain a series from which e may be evaluated:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

The second is linked to compound interest. If one is faced with the question, do you wish to have bank interest added at the rate of 100% once a year, the rate of 50% twice a year, 10% ten times a year, or whatever you fancy based on this type of formula, then what is the best option? Well, if one has interest added n times in the year, the amount of money you will have at the end of the year will have increased by a factor of,

$$F = \left(1 + \frac{1}{n}\right)^n.$$

It turns out that F increases as n increases, and the limiting case of microscopic rates being added infinitely often yields $F = e$. We will prove this later in the unit using l'Hôpital's rule.

The value, e, is also the base for the natural logarithm: $y = \ln x$ means that $x = e^y$.

The exponential function. As x increases, it rises faster than any power of x. As x becomes increasingly negative, it decreases faster than any inverse power of x.

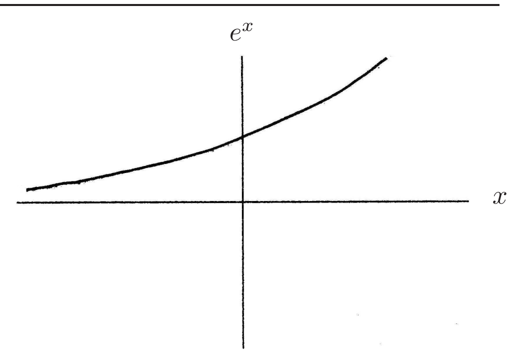

Figure 1.18.

The decreasing exponential function. This is simply the reciprocal of e^x.

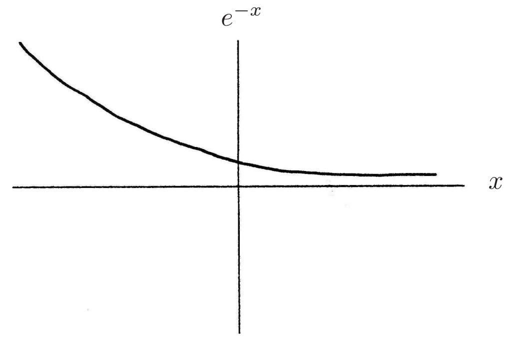

Figure 1.19.

Welcome to your first hyperbolic function. It is defined as $\cosh x = (e^x + e^{-x})/2$ and so it is an 'average' of the previous two Figures. It is the hyperbolic counterpart to $\cos x$ in that it is sometimes referred to as the hyperbolic cosine. We usually say 'cosh x' — cosh, as in the weapon used by 19th century miscreants.

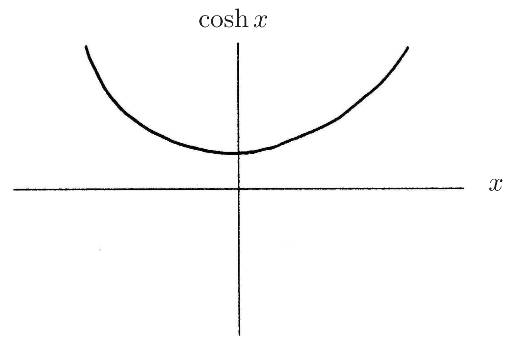

Figure 1.20.

Welcome to your second hyperbolic function. It is defined as $\sinh x = (e^x - e^{-x})/2$ and so it is half the difference between the curves in Figures 1.18 and 1.19. It is the hyperbolic counterpart to $\sin x$ in that it is sometimes referred to as the hyperbolic sine. For this one we usually say 'shine x'.

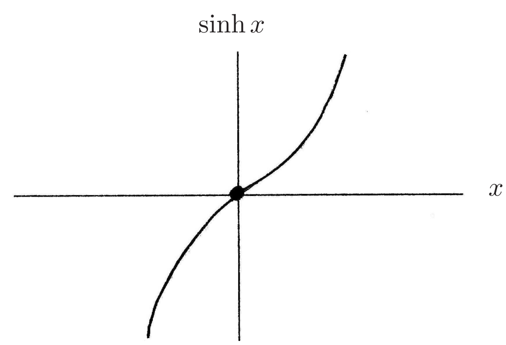

Figure 1.21.

This is the quotient of $\sinh x$ and $\cosh x$. Although written as $\tanh x$ we say, 'tanch x'. It is, of course, the hyperbolic tangent function. Given that

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}},$$

it is clear that $\tanh x \to 1$ when $x \to \infty$. A similar argument leads to $\tanh x \to -1$ when $x \to -\infty$. It's a much nicer function than $\tan x$ because there are no infinities!

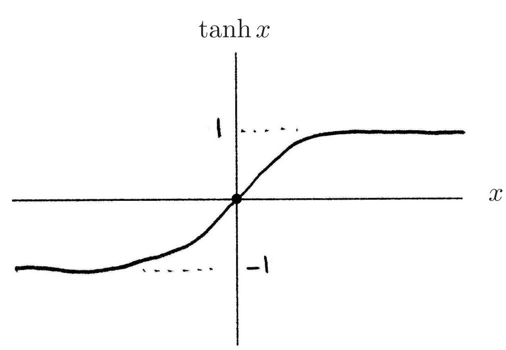

Figure 1.22.

Another nice-looking function. This one arises in Normal Probability Distributions. Hopefully it is clear to see why this decays when $x \to \pm\infty$. Strictly speaking, this function doesn't decay exponentially, but it does so super-exponentially, which is considerably faster.

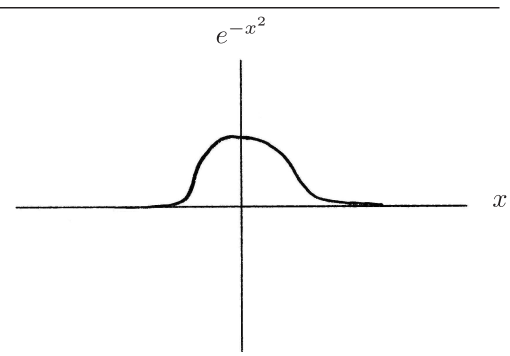

Figure 1.23.

We are now entering the realms of the crazy. First, we must consider the exponent — how does $1/x^2$ behave? Well, when $x \to \infty$, then $1/x^2 \to 0$ and hence $\exp(1/x^2) \to 1$. The same goes for when $x \to -\infty$. When $x \to 0$, then $1/x^2 \to \infty$ and hence $\exp(1/x^2) \to \infty$. All of this may be seen in the Figure.

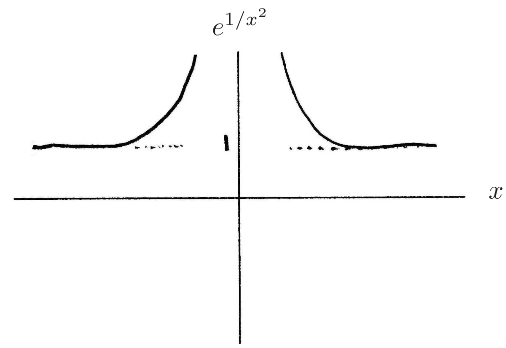

Figure 1.24.

The reciprocal of the function in Figure 1.24, we may mentally find that reciprocal and then draw it. Alternatively, we could follow the same type of argument as above. You may be interested to know that not only is this function equal to zero when $x = 0$, but all of its derivatives are too. This is exceptionally flat.

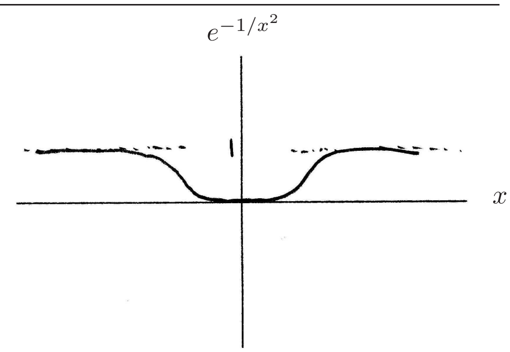

Figure 1.25.

1.5 Comments on symmetries.

We'll take a breather at this point and consider symmetries. While geometrical shapes may have symmetries (e.g. a square has four axes of reflective symmetry, and three different rotational symmetries), mathematical functions may also be said to have symmetries. These come in three main kinds, namely (i) symmetric (i.e. even), (ii) antisymmetric (i.e. odd) and (iii) asymmetric (i.e. no symmetry — the prefix, a, means 'without', as in the words, atheist, amoral, amuse, apnea.

An even function is one which has mirror symmetry in the vertical axis. Figures 1.2, 1.9, 1.15 and 1.20 are examples of even functions, although there are more above. Mathematically we may say that an even function satisfies $f(-x) = f(x)$. Examples of this include, $\cos(-x) = \cos(x)$, and $(-x)^2 = x^2$.

The easiest pictorial way of describing an odd function is to say that it stays the same when one rotates it through $180°$ about the origin. Figures 1.1, 1.6, 1.7, 1.8, 1.21 and 1.22 are all odd. Mathematically we have $f(-x) = -f(x)$ for odd functions. Let us try it out: $\sin(-x) = -\sin(x)$ and $(-x)^3 = -x^3$, therefore we conclude that both $\sin x$ and x^3 are odd. Occasionally one must be careful, though: $|\sin(-x)| = |-\sin(x)| = |\sin(x)|$, shows us that $|\sin x|$ is even, as seen in Figure 1.17. The same sort of argument tells us that both $\sin^2 x$ and $\sin(x^2)$ are even.

The third type is asymmetric, and so the function is neither even nor odd. Figures 1.4 and 1.14 are examples of this. The simple addition of an even function and an odd function guarantees the generation of an asymmetric function: $x + x^2$ is an easy example.

One aspect which will be used very soon is to consider what happens when one multiplies two functions together. We know that both x and x^3 are odd, but their product, x^4, is even. This is indeed a general property: odd×odd=even. By taking other simple examples, we may see that even×even=even and that even×odd=odd are also true in general.

Finally, if is important to note that our definitions been made with regard to the properties of the function relative to $x = 0$. We could say that Figures 1.4 and 1.5 are both even about $x = 1$. It is also possible to state that, while $\sin x$ is an odd function, it is nevertheless even about $x = \pi/2$, and asymmetric about $x = \pi/4$.

1.6 Envelopes.

This class of function generally involves the product of two functions, one of which oscillates. We will look at the functions, $x\cos x$, $x\sin x$, $(\cos x)/x$ and $e^{-|x|}\sin 100x$.

As a preamble to presenting examples, let us consider the general case of $f(x)\sin x$. We can imagine $f(x)$ to be a fairly simple function, although it is not advisable at first to think of a sinusoidal function. Given that $\sin x$ lies between -1 and 1, the function, $f(x)\sin x$, must lie between $\pm|f(x)|$ everywhere. Therefore a good strategy for sketching envelopes would to draw both $f(x)$ and $-f(x)$ first, then mark in where $\sin x = 0$, check any possible symmetries that might apply (see below for specific examples), and finally attempt to sketch the curve.

We first mark in the functions x and $-x$ as the dashed lines, and therefore we know that $x\cos x$ must lie between these two extremes. The zeros of $\cos x$ are at $x = \pm\pi/2$, $\pm 3\pi/2$ and so on, so these are marked by black disks on the x-axis. Another zero arises at $x = 0$ due to the factor, x, in $x\cos x$. Finally, we note that the function is odd (odd×even).

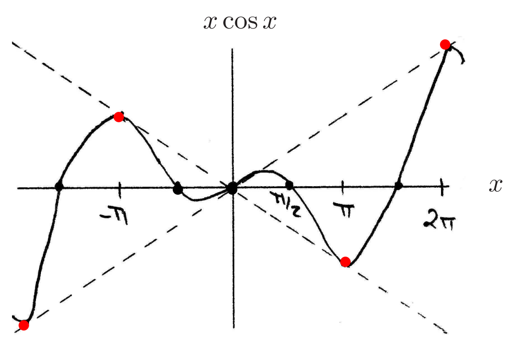

Figure 1.26.

This example is very similar to that shown in Figute 1.26. However, the function is even because $x\sin x$ is odd×odd. While $\sin x$ has single zeros at $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$ and so on, there is a double zero at $x = 0$ because both the factors in $x\sin x$ have simple zeros there. Therefore the curve will look like a parabola near $x = 0$.

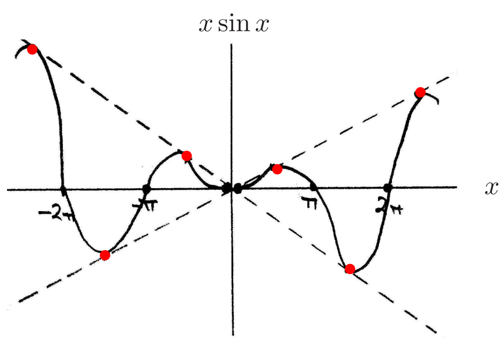

Figure 1.27.

We first mark in the functions $1/x$ and $-1/x$. These tend to plus or minus infinity as $x \to 0$. This is an odd function and the sketch follows after finding where the zeros of $\cos x$ are, and knowing that both $\cos x$ and x are positive when x is slightly greater than zero.

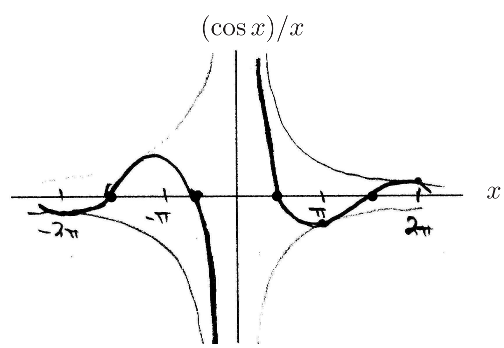

Figure 1.28.

The function $e^{-|x|}$ decays exponentially as $x \to \infty$ and as $x \to -\infty$. It is even. We have plotted $\pm e^{-|x|}$ to act as the guide lines, i.e. the envelope. The odd function $\sin 100x$ oscillates very quickly (the period is $\pi/50$). Hence $e^{-|x|}\sin 100x$ is an odd function because it is the product of one even and one odd function. This is an example of an AM radio signal: the carrier wave ($\sin 100x$) has its amplitude modulated by the signal, $e^{-|x|}$.

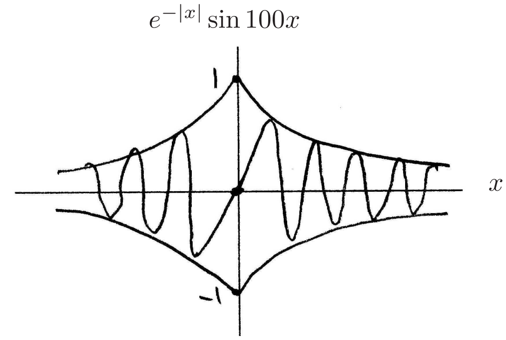

Figure 1.29.

1.7 Square roots of functions.

A little bit of care needs to be taken whenever one needs to sketch the square root of a function. The question is always whether one needs to take both square roots, although that question is often resolved when considering what the application is. Of more importance is what happens when one takes the square root of a function which has a single zero. It is best to illustrate this by plotting $y = \sqrt{x}$.

We have taken the positive root only in this sketch. If we square both sides of $y = \sqrt{x}$, then we get $y^2 = x$, which is a parabola, but one which is rotated through 90° compared with the one in Figure 1.2. Thus the slope of the present graph at the origin is infinite, i.e. the tangent to the curve is a vertical line. This will always be true when encountering the square root of a linear factor at the value of x where the factor is zero.

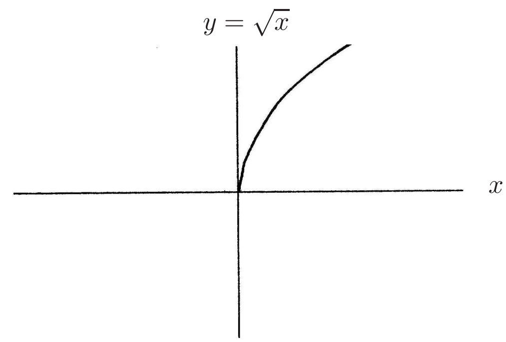

Figure 1.30.

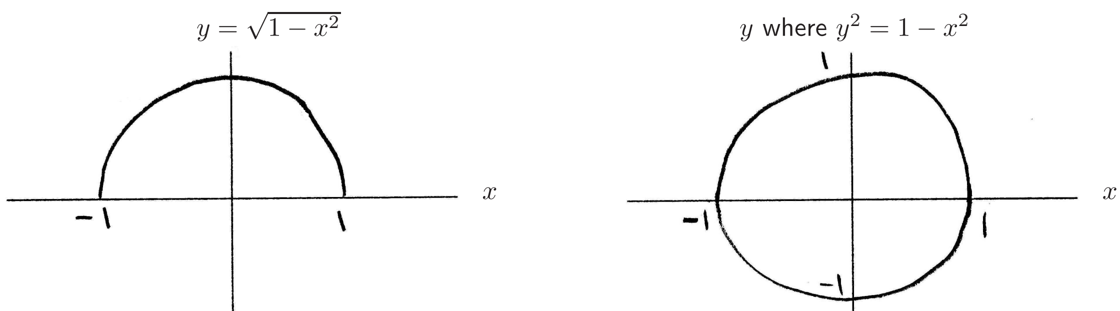

Figure 1.31.

Here I am simply making the distinction between writing $y = \sqrt{1 - x^2}$ explicitly, and writing $y^2 = 1 - x^2$ in terms of what one plots. In the right hand graph the negative values of y are allowed, whereas they are not allowed on the left.

That this is a circle follows from rearranging $y^2 = 1 - x^2$ into the form $x^2 + y^2 = 1$. Obviously the circle is a well-known shape, and therefore we are not surprised that there is a vertical tangent (or infinite slope) at $x = 1$. But this also follows from the short discussion on Figure 1.30, as will now be explained. For the present equation, namely, $y = \sqrt{1 - x^2}$, we may factorise it into the form, $y = \sqrt{(1 + x)(1 - x)}$. When x is close to 1 but slightly below it, then $(1 + x) \simeq 2$ and therefore the equation becomes $y \simeq \sqrt{2(1 - x)}$. This is the square root of a linear factor, just as we have displayed in Figure 1.30.

This is the square root of the function which was plotted in Figure 1.9. That function has two intervals in which it is negative and therefore the square roots doesn't exist, at least for the purposes of sketching here. That function has four simple zeros, and therefore we expect the square root of that function to look like sideways parabolae at those points. We also note that $y \simeq x^2$ when $x \to \pm\infty$. The dotted lines yield the negative part of the sketch that I would have produced should I have asked for

$$y^2 = (x-2)(x-1)(x+1)(x+2).$$

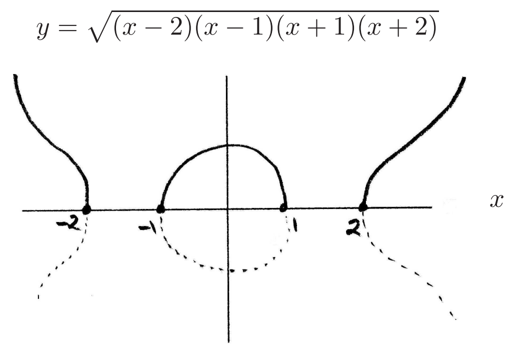

Figure 1.32.

1.8 Ratios of polynomials.

This is the final topic. It sounds scary, but this is very systematic. An example might be

$$y = \frac{(x-1)^2(x+2)}{x(x-3)^2}.$$

The general set of information that one must collect at the outset is: (i) the locations of all zeros and their multiplicity (i.e. single, double and so on); (ii) the locations of all poles (i.e. infinities) which are the zeros of the denominator, and we also need their multiplicity; (iii) the large-x behaviour. For the above example we have the following information:

Zeros at $x = 1, 1, -2$ i.e. $x = 1$ twice and $x = -2$ once,

Infinities at $x = 0, 3, 3$ i.e. $x = 3$ twice and $x = 0$ once,

$y \to 1$ as $x \to \pm\infty$.

This last piece of information comes from the observation that, when x is large (either positively so or negatively) then the constants are negligible, and therefore $y \simeq x^3/x^3 = 1$.

We won't sketch this particular one, but it is important to know that the 'infinities' are also known as **singularities**. More precisely we would say that $x = 0$ is either a **simple pole** or a **first order pole**, and that $x = 3$ is either a **double pole** or a **second order pole**.

A straightforward case of a pole at $x = 0$. As x increases past $x = 0$, the sign of y changes, but y will have descended down to $-\infty$ before re-emerging at $+\infty$. This infinite discontinuity always happens at a single pole.

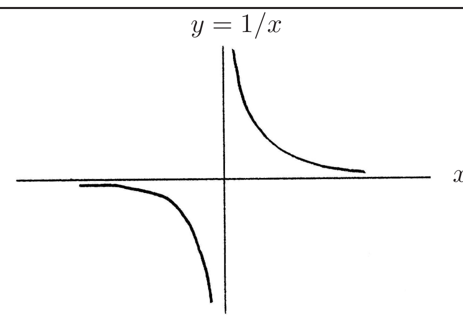

Figure 1.33.

As Figure 1.33 but the pole is now at $x = 1$.

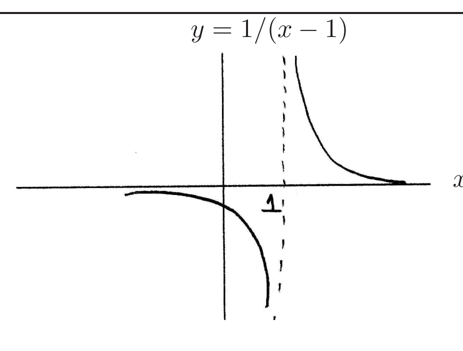

Figure 1.34.

This is a double pole at $x = 1$. I have indicated the location of the double pole by a pair of dashed lines. The curve's shape is qualitatively different from that of the single pole shown in Figure 1.34. The presence of the square means that the function is always positive, but more importantly we need to note that as x increases past $x = 1$, then the sign does not change. Hence the curve ascends to $+\infty$ and then returns from $+\infty$.

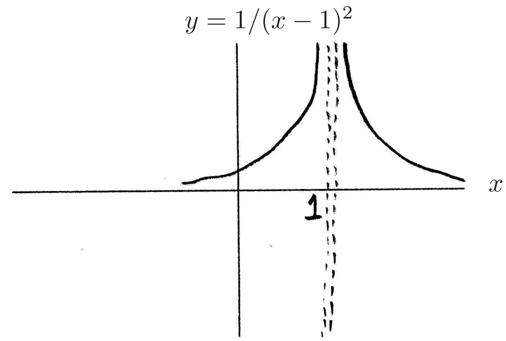

Figure 1.35.

This is a triple pole at $x = 1$, the presence of which is indicated by three dashed lines. The fact that it is a cube means that there will be a sign change as x increases past $x = 1$. So this looks very similar to the single pole case.

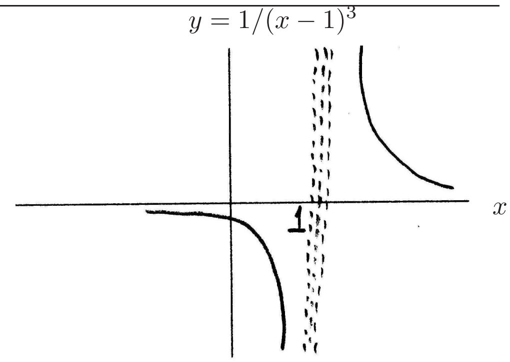

Figure 1.36.

$y = \dfrac{1}{x^2 - 1}$.

There are no zeros.

Poles at $x = -1$ and $x = 1$
because $x^2 - 1 = (x-1)(x+1)$.

When $x \to \pm\infty$ then $y \to 0$.

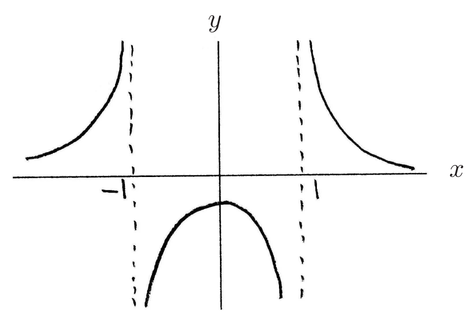

Figure 1.37.

Beginning at a large positive value of x we note that $y > 0$. Therefore the curve must rise from the x-axis at first. As x approaches $x = 1$ from above, y must rise all the way to infinity and re-emerge from $-\infty$ because of the single pole at $x = 1$. As x decreases further towards the second pole at $x = -1$, the curve must turn back downwards towards $-\infty$; it cannot travel through the x-axis because there is no zero between $x = -1$ and $x = 1$ — this is the only option for the curve. On the negative side of $x = -1$ the curve re-emerges from $+\infty$ and descends towards $y = 0$, the large-$|x|$ value.

All of our examples will follow a similar logic. The behaviour of the curve will then depend on the multiplicity of the zeros and poles.

$y = \dfrac{x}{x^2 - 1}$.

Zero at $x = 0$.

Poles at $x = -1$ and $x = 1$.

When $x \to \pm\infty$ then $y \to 0$.

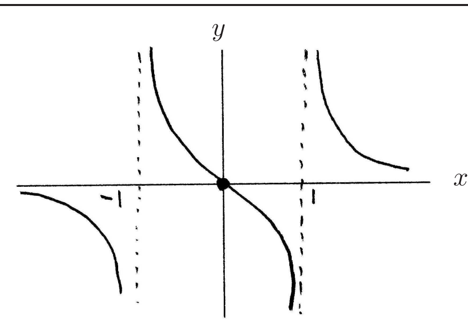

Figure 1.38.

The logic begins in the same way as for Figure 1.37 except that there is now a zero in-between $x = 1$ and $x = -1$. Therefore the curve must pass through that zero and continue to ascend to $+\infty$ at $x = -1$. It is worth comparing Figures 1.37 and 1.38 to see the difference caused by the addition of a zero.

$y = \dfrac{x(x-1)}{(x+1)(x-2)}$.

Zeros at $x = 0, 1$.

Poles at $x = -1, 2$.

When $x \to \pm\infty$ then $y \to 1$.

Similar logic to the previous example.

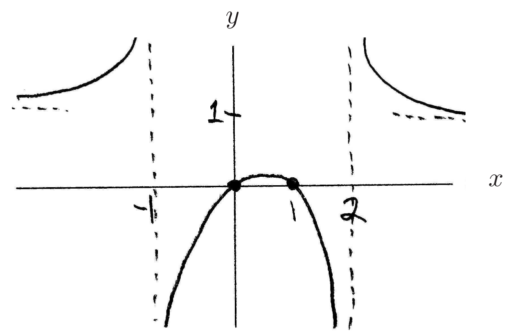

Figure 1.39.

$y = \dfrac{x(x-1)^2}{(x+1)^2(x-2)}$.

Zeros at $x = 0, 1, 1$.

Poles at $x = -1, -1, 2$.

When $x \to \pm\infty$ then $y \to 1$.

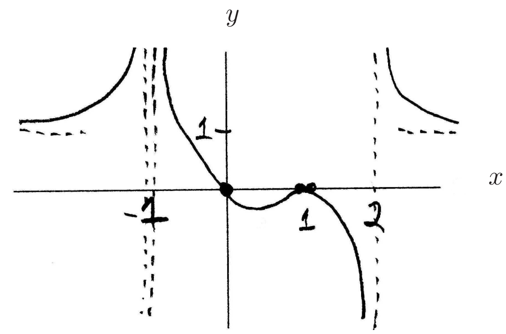

Figure 1.40.

This example has both a double zero (at $x = 1$) and a double pole (at $x = -1$). Therefore the curve resembles a parabola near to $x = 1$, a downward-facing one. The double pole has the same feature as the one that was discussed for Figure 1.35, namely that the sign doesn't change.

$y = \dfrac{x(x-1)^2}{(x+1)(x-2)}$.

Zeros at $x = 0, 1, 1$.

Poles at $x = -1, 2$.

When $x \to \pm\infty$ then $y \sim x$.

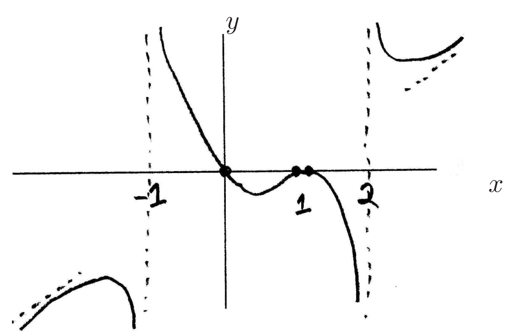

Figure 1.41.

A slightly modified version of both Figures 1.39 and 1.40. Note the different large-x behaviour here.

Final note.

There are some further examples of the sketching of ratios of polynomials available from the Handouts section of the unit webpage. This includes a detailed account of quite a complicated example. The direct link, should you wish to type it, is

`http://staff.bath.ac.uk/ensdasr/ME10304.bho/extra.curves.pdf`

There is some extra information in the Miscellaneous section of the unit webpage on hyperbolic functions. There's a little bit of calculus, some hyperbolic versions of the multiple angle formulae, some graphs as opposed to sketches, and an explanation by way of differential equations why there is a similarity between the hyperbolic and the trigonometric functions. The direct link is,

`http://staff.bath.ac.uk/ensdasr/ME10304.bho/hyperbolics.pdf`