1 Curve sketching — Polynomials, moduli, exponentials and hyperbolics

1.1 Preamble

- The chief aim of this short section of the Maths 1 unit is to develop one's understanding of what shapes the various standard functions have so that more complicated functions may be sketched or identified.
- We will attempt to sketch functions by looking at the detailed formula for that function.
- We'll begin with polynomials and then complicate issues with the modulus function, exponentials, envelopes, square roots, and finally ratios of polynomials.
- We will not employ any methods from calculus in this section.
- We will ask questions such as, where are the zeroes, where are the asymptotes, what is the large-x behaviour?

1.2 Polynomials

Examples of polynomials:

$$y = x^2 \qquad\qquad y = x^3 - x \qquad\qquad y = (x-5)^2(x-1)x$$

$$y = x(x-1)(x-2)(x-3)(x-4)(x-5).$$

Much will be determined by knowing where the roots are, i.e. those values of x for which $y(x) = 0$.

We will start from the utterly trivial (e.g. a linear function) and make our way to slightly more complicated shapes (quadratics, cubics and quartics).

The straight line, $y = x$, should be quite straightforward! The bullet indicates the single zero or root.

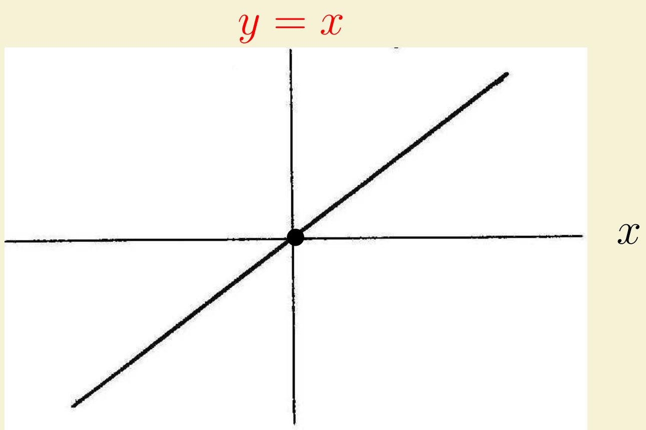


Figure 1.1.

This is a parabola, of course. It has a zero slope at $x = 0$. From the point of view of factorization, $y = x \times x$, we may say that it also has a 'double zero' at $x = 0$. This will be important below. The bottom of the parabola (double bullet) shows what a double zero looks like.

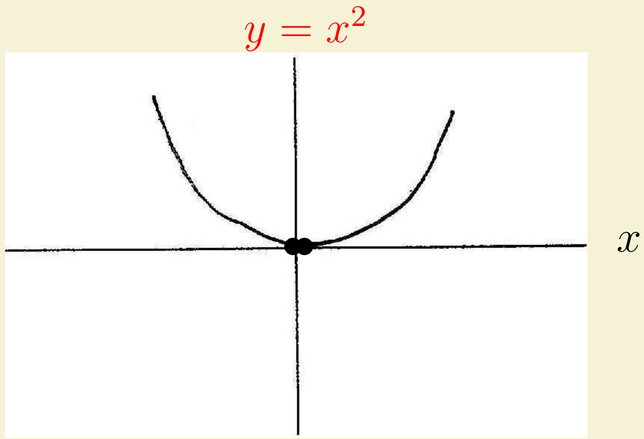


Figure 1.2.

This is a downward-facing parabola. All parabolae, $y = ax^2 + bx + c$, will look like this when $a < 0$, although the maximum will not be at the origin in general.

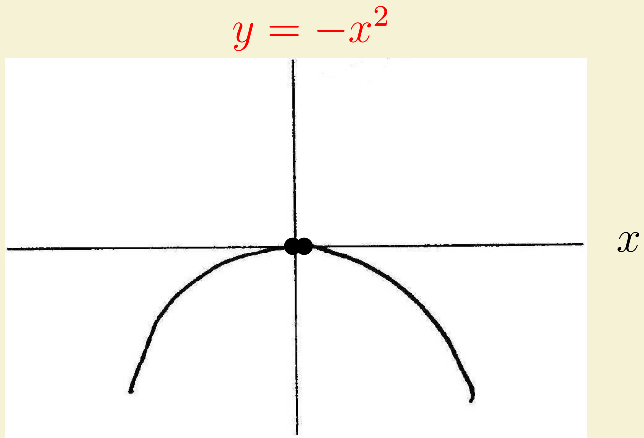


Figure 1.3.

This parabola is identical to the one in Figure 1.2, but has been shifted by 1 in the x-direction. We see that there is a double root at $x = 1$ — note the repeated $(x-1)$ factor.

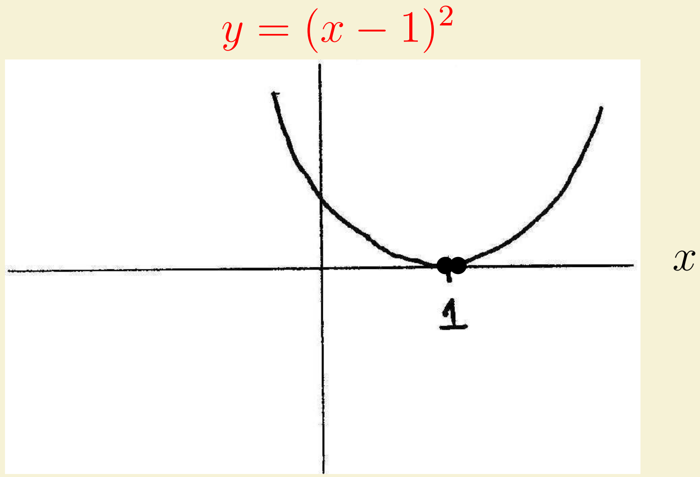


Figure 1.4.

This parabola is identical to the one in Figure 1.4, but it has been shifted downwards by 1. The resulting expression for y may be factorised into $y = x(x-2)$ and hence there are two zeros: $x = 0$ and $x = 2$.

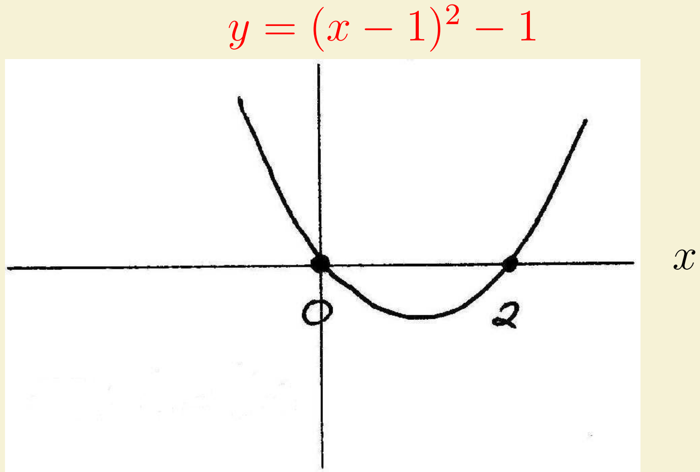


Figure 1.5.

Cubic functions

The general for is $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$.

Three varieties:

- (i) with a local maximum and a local minimum;
- (ii) with a point of inflexion;
- (iii) without a maximum, minimum or point of inflexion.

This cubic has a local maximum and a local minimum. Given that it may be factorised into the form, $y = x(x-1)(x+1)$, it has the three roots, $x = -1, 0, 1$. Note that cubics of this type don't always have three roots: if this cubic were moved upwards by 100 to give, $y = x^3 - x + 100$, then there would only be one root.

$y = x^3 - x$

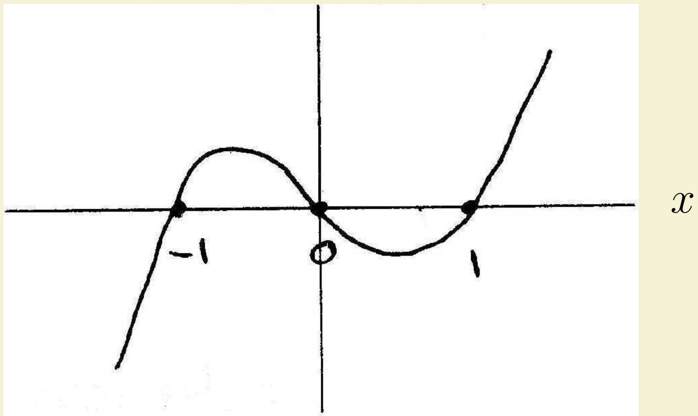


Figure 1.6.

This is the standard cubic function. It has a point of inflexion at $x = 0$, which means that both the slope and the second derivative are zero there. Given that we may write this as $y = x \times x \times x$, we see that a point of inflexion sitting on the horizontal axis corresponds to a triple zero.

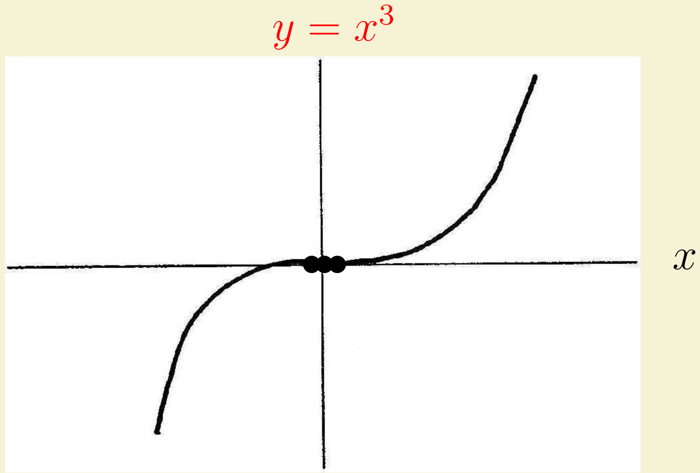


Figure 1.7.

Cubics such as this have no critical points, by which I mean maxima, minima or points of inflexion. They always have one zero, though. In this case it is at $x = 0$ because $y = x(1 + x^2)$.

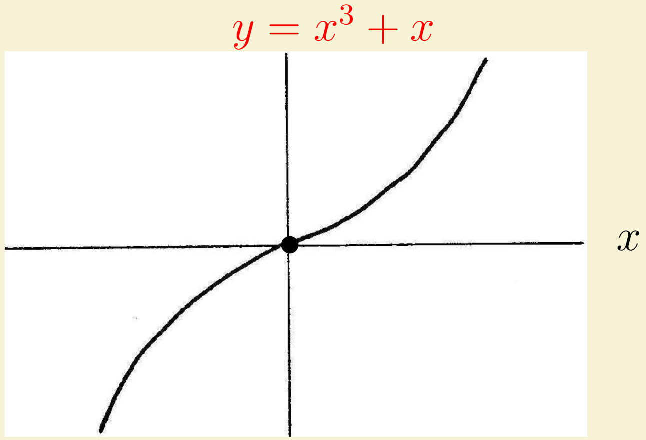


Figure 1.8.

Quartic functions

Note the difference between the words, *quadratic* and *quartic*.

The general form is $y = ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$. Given the number of constants,

Alternatively: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = \displaystyle\sum_{n=0}^{4} a_n x^n$.

More generally for an N^{th} order polynomial:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{N-1} x^{N-1} + a_N x^N = \sum_{n=0}^{N} a_n x^n.$$

Quartics come in the following varieties:

- (i) two maxima and one minimum (or vice versa);
- (ii) a point of inflexion and either a maximum or a minimum;
- (iii) a quartic minimum (or maximum);
- (iv) a standard parabolic minimum or maximum.

This quartic curve has three extrema: two minima and one maximum. It also has four zeros, namely $x = -2$, -1, 1 and 2, which may be found directly from the function itself. Note that if we were to add exactly the right constant (it turns out to be $9/4$) to this function in order to raise the minima so that they both lie on the x-axis, then we would now have a pair of double zeros.

$$y = (x-2)(x-1)(x+1)(x+2)$$

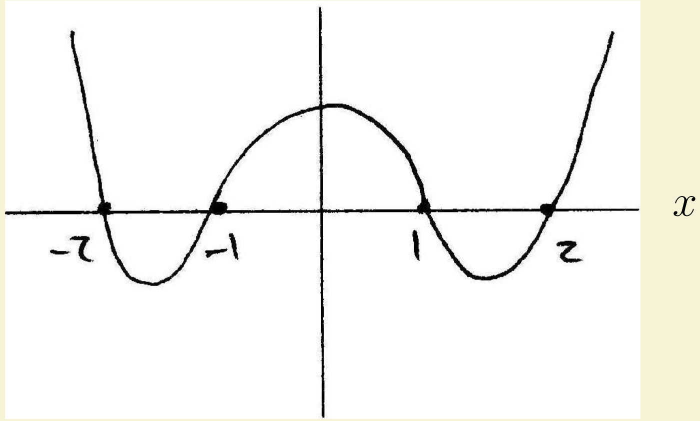


Figure 1.9.

This curve has four zeros: $x = 0$, 0, 0 and 2. Therefore there is a point of inflexion at $x = 0$ (i.e. a triple zero) and a simple zero at $x = 2$.

$$y = (x-2)x^3$$

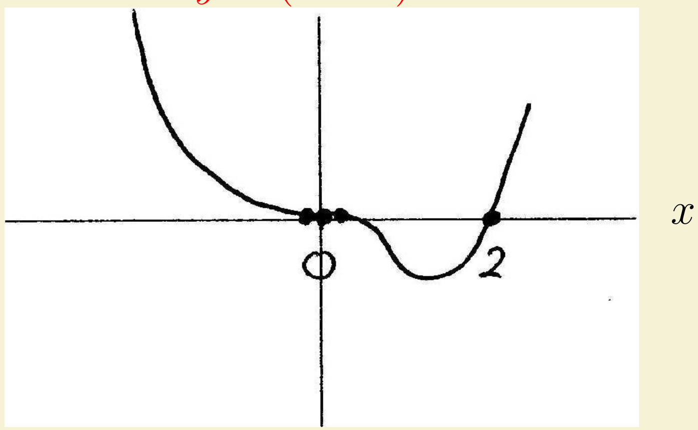


Figure 1.10.

The pure quartic function: $y = x^4$. The first three derivatives are zero at $x = 0$ and therefore this curve has a much flatter base than the parabola does, and this must be shown clearly in the sketch. We also have a four-times repeated root, and therefore $x = 0$ is a **quadruple zero**.

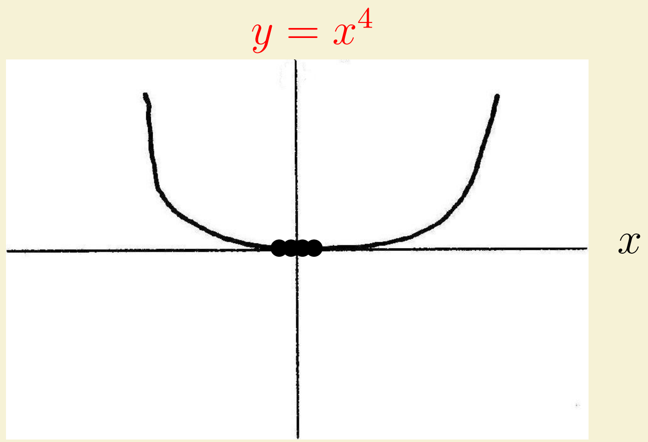


Figure 1.11.

This looks like a parabola because the x^2 term is much larger than x^4 is when x is small. The value, $x = 0$, corresponds to a double zero because $x^4 + x^2 = x^2(x^2 + 1)$. However, the function grows much faster as x increases than a parabola does because of the x^4 term, although this is quite difficult to show on a simple sketch.

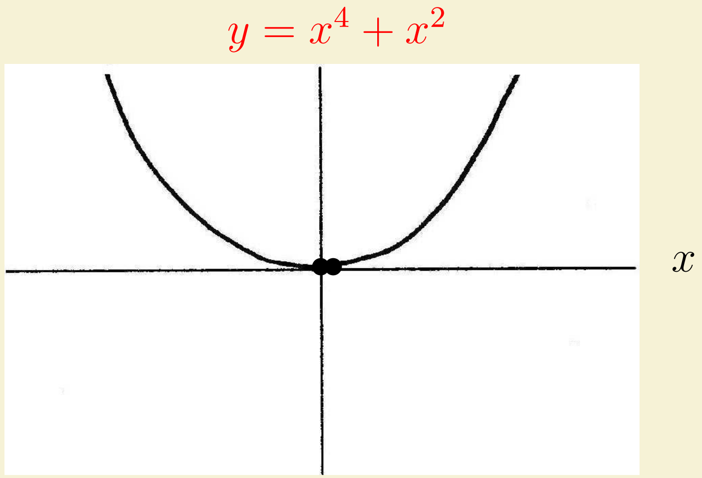
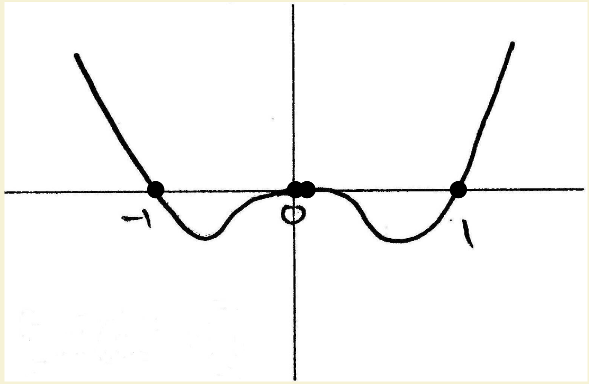


Figure 1.12.

This quartic curve is very similar to that displayed in Figure 1.9. While the general shape is the same, this one has single zeros at $x = \pm 1$ and a double zero at $x = 0$ because $x^4 - x^2 = x^2(x+1)(x-1)$.

$y = x^4 - x^2$

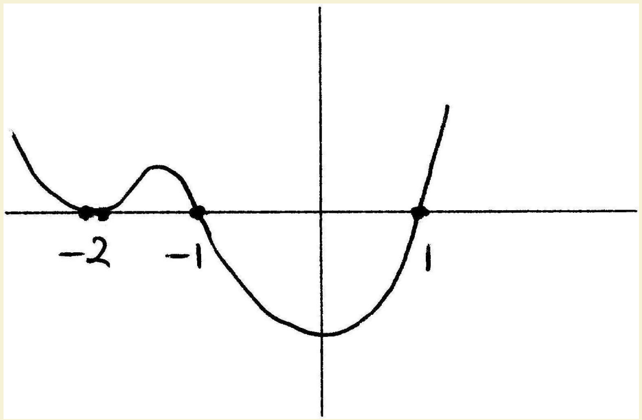


x

Figure 1.13.

This final quartic curve has been inferred from the given factorisation. There is a double zero at $x = -2$ and single zeros at $x = -1$ and $x = 1$. Given that the coefficient of x^4 is positive, the function becomes large and positive when $x \to \infty$. The sketch of $-(x^2-1)(x+2)^2$ is the present one turned upside-down.

$y = (x^2 - 1)(x+2)^2$



x

Figure 1.14.

1.3 Moduli

The modulus of a function is its absolute numerical value. So $|5| = 5$ and $|-5| = 5$.

We may write this mathematically as follows:

$$|f(x)| = f(x) \quad \text{when} \quad f(x) \geq 0, \qquad\qquad |f(x)| = -f(x) \quad \text{when} \quad f(x) \leq 0.$$

We could also define it as the positive square root of the square of the function:

$$|f(x)| = +\sqrt{[f(x)]^2}.$$

So $|f(x)| \geq 0$. Colloquially we speak of $|f(x)|$ as 'mod f' or 'mod f of x'.

We see that $|x|$ is positive everywhere except at the origin where it is zero.

$y = |x|$

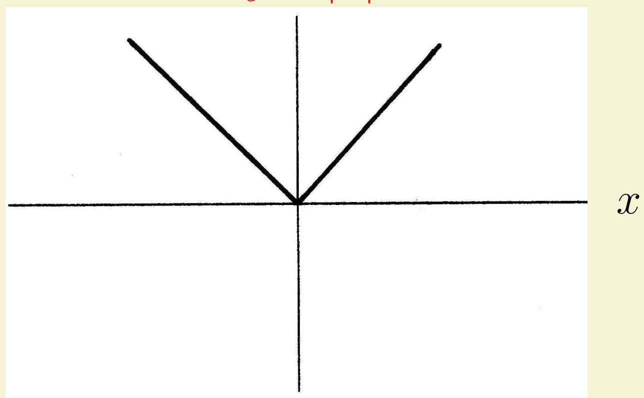


Figure 1.15.

This quartic curve has three extrema: two minima and one maximum. It also has four zeros, namely $x = -2$, -1, 1 and 2, which may be found directly from the function itself. Note that if we were to add exactly the right constant (it turns out to be $9/4$) to this function in order to raise the minima so that they both lie on the x-axis, then we would now have a pair of double zeros.

$y = (x-2)(x-1)(x+1)(x+2)$

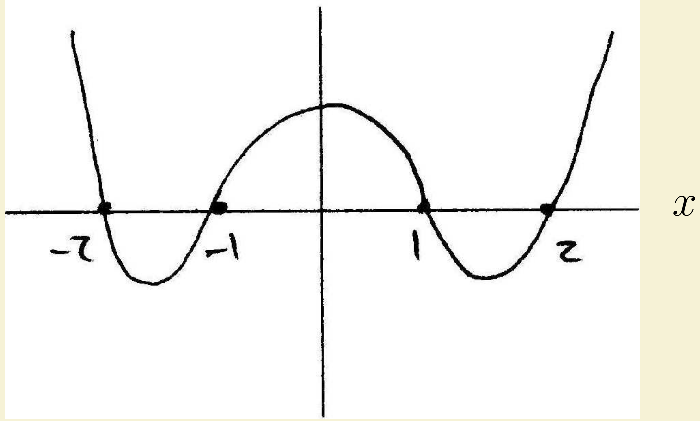


Figure 1.9.

This is 'modulus' version of Figure 1.9. I have constructed this by drawing the curve given in Figure 1.9 using short dashes. Those values which are positive have been overdrawn with a continuous line, while those parts which are negative have been multiplied by -1 and then they are drawn in with a continuous line.

$y = |(x-2)(x-1)(x+1)(x+2)|$

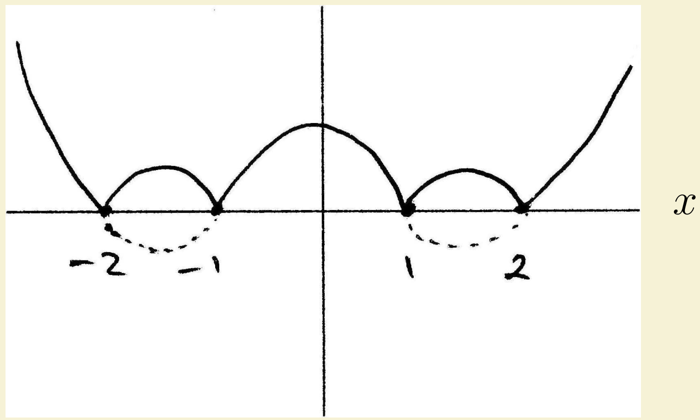


Figure 1.16.

This has been constructed in the same way as for the previous Figure. For this particular function, we may also refer to it as a 'rectified sine wave' — this is often used in Electrical Engineering. While $\sin x$ has a period of 2π, $\sin|x|$ has a period of π.

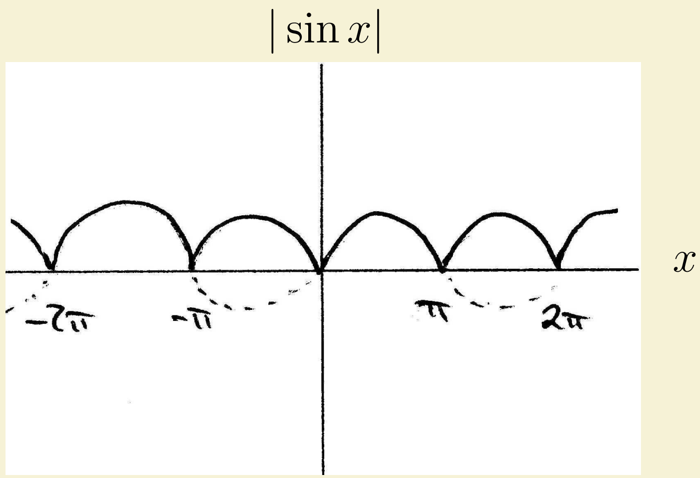


Figure 1.17.

I'll leave this as an exercise, but is $\sin|x|$ the same as $|\sin x|$?

1.4 Exponential and hyperbolic functions.

The exponential function is typified by e^x where $e = 2.7182818284590452$ to 16 decimal places. This strange number arises in many places. One of the most common is the following power series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

a result which we will prove later in the unit, but if we set $x = 1$ we obtain a series from which e may be evaluated:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots .$$

The second is linked to compound interest. If one is faced with the question, do you wish to have bank interest added at the rate of 100% once a year, the rate of 50% twice a year, 10% ten times a year, or whatever you fancy based on this type of formula, then what is the best option? Well, if one has interest added n times in the year, the amount of money you will have at the end of the year will have increased by a factor of,

$$F = \left(1 + \frac{1}{n}\right)^n.$$

It turns out that F increases as n increases, and the limiting case of microscopic rates being added infinitely often yields $F = e$. We will prove this later in the unit using l'Hôpital's rule.

The value, e, is also the base for the natural logarithm: $y = \ln x \Rightarrow x = e^y$, but only when $x > 0$.

The exponential function. As x increases, it rises faster than any power of x. As x becomes increasingly negative, it decreases faster than any inverse power of x.

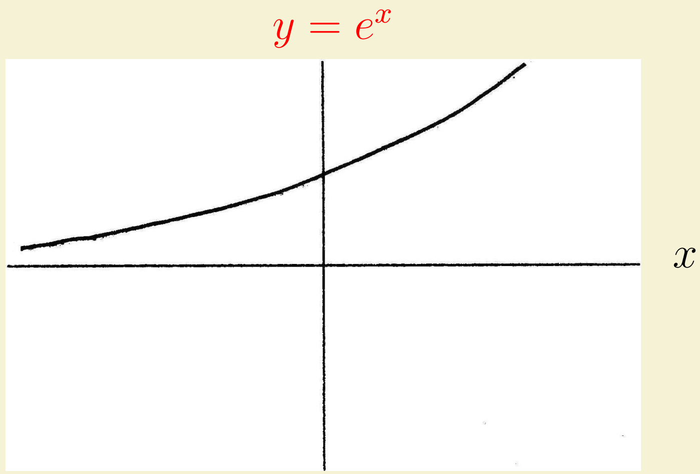


Figure 1.18.

The decreasing exponential function. This is simply the reciprocal of e^x.

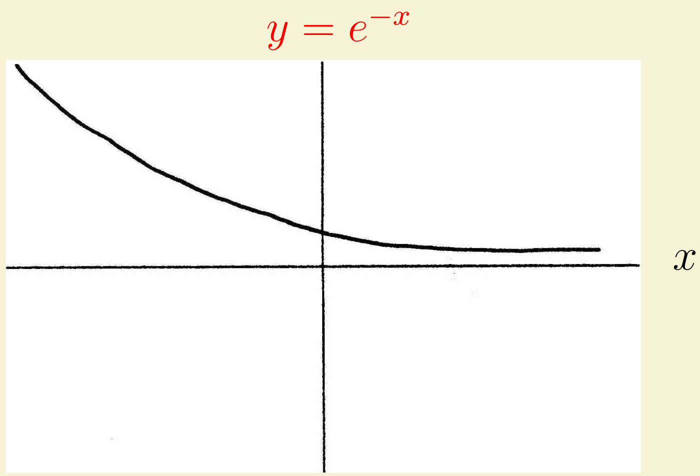


Figure 1.19.

Welcome to your first hyperbolic function: $\cosh x = (e^x + e^{-x})/2$ and so it is an 'average' of the previous two Figures. It is the hyperbolic counterpart to $\cos x$ in that it is sometimes referred to as the hyperbolic cosine. We usually say 'cosh x' — cosh, as in the weapon used by 19th century miscreants.

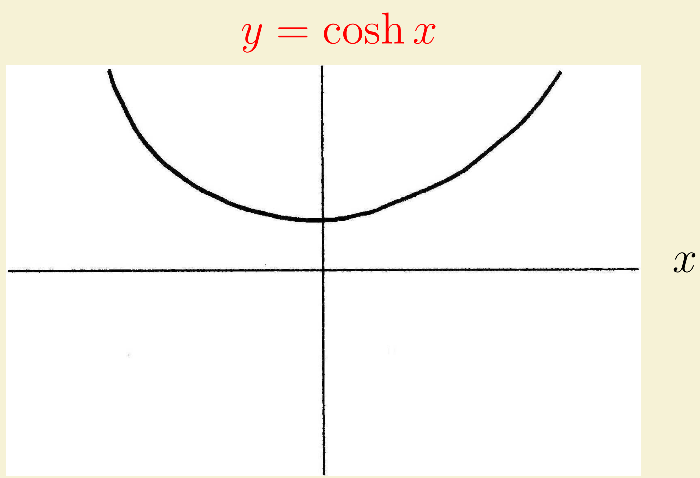


Figure 1.20.

Welcome to your second hyperbolic function: $\sinh x = (e^x - e^{-x})/2$ and so it is half the difference between the curves in Figures 1.18 and 1.19. It is the hyperbolic counterpart to $\sin x$ in that it is sometimes referred to as the hyperbolic sine. For this we usually say 'shine x'.

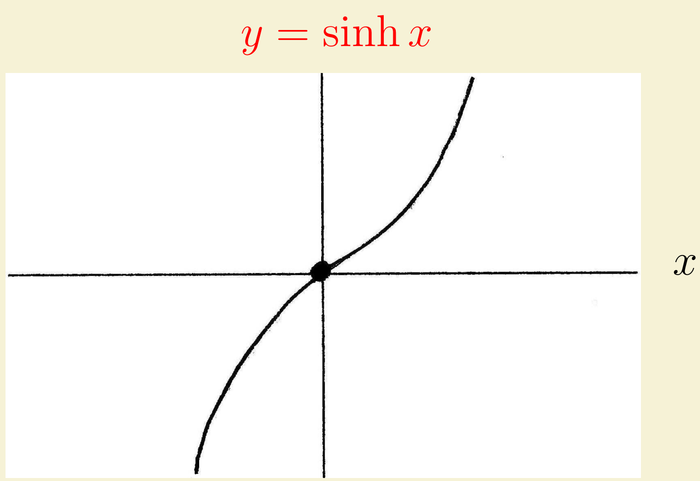


Figure 1.21.

This is the quotient of $\sinh x$ and $\cosh x$. Written as $\tanh x$ we say, 'tanch x'. It is, of course, the hyperbolic tangent function. Given that

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}},$$

it is clear that $\tanh x \to 1$ when $x \to \infty$. A similar argument leads to $\tanh x \to -1$ when $x \to -\infty$. It's a much nicer function than $\tan x$ because there are no infinities!

$y = \tanh x$

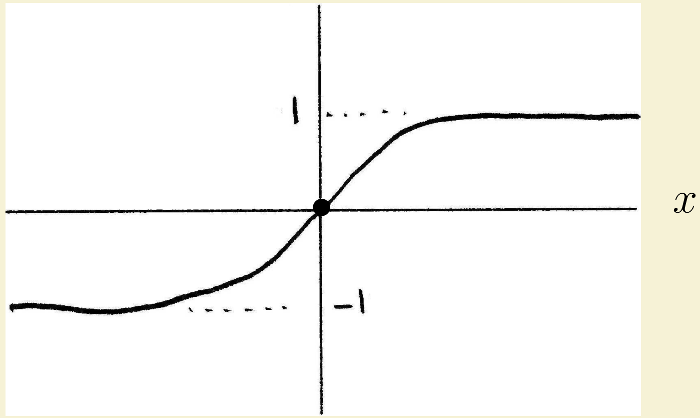


Figure 1.22.

Another nice-looking function. This one arises in Normal Probability Distributions. Hopefully it is clear to see why this decays when $x \to \pm\infty$. Strictly speaking, this function doesn't decay exponentially, but it does so super-exponentially, which is considerably faster.

$y = e^{-x^2}$

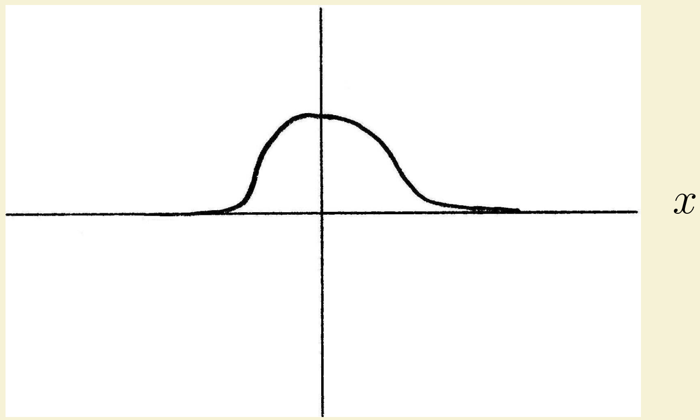


Figure 1.23.

We are now entering the realms of the crazy. First, we must consider the exponent — how does $1/x^2$ behave? Well, when $x \to \infty$, then $1/x^2 \to 0$ and hence $\exp(1/x^2) \to 1$. The same goes for when $x \to -\infty$. When $x \to 0$, then $1/x^2 \to \infty$ and hence $\exp(1/x^2) \to \infty$. All of this may be seen in the Figure.

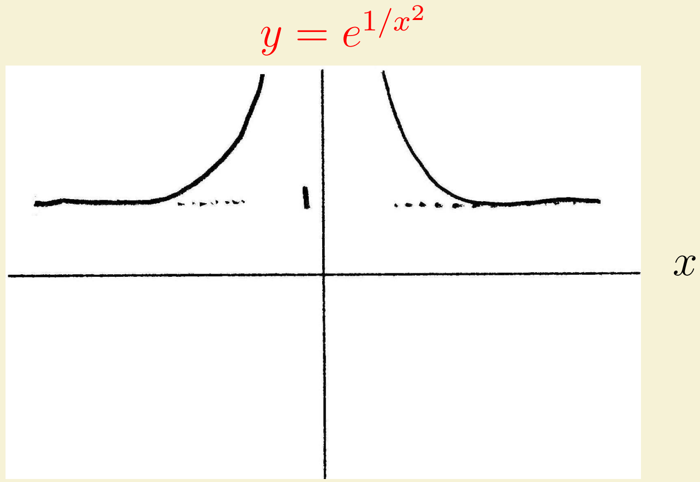


Figure 1.24.

The reciprocal of the function in Figure 1.24, we may mentally find that reciprocal and then draw it. Alternatively, we could follow the same type of argument as above. You may be interested to know that not only is this function equal to zero when $x = 0$, but all of its derivatives are too. This is exceptionally flat.

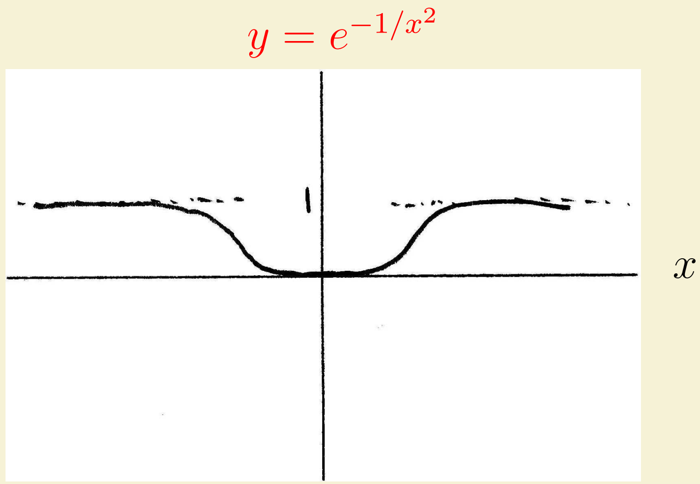


Figure 1.25.