9/10/2020

Department of Mechanical Engineering, University of Bath

Mathematics 1 ME10304

Problem Sheet 3 — Differentiation

Q1. Use the 'small increment' method (i.e. the approach using limits) described in the lecture notes to find the derivative of the following functions: [NOTE: **(i)** this type of question will *not* appear in the exam; **(ii)** part (d) begs the question – you'll see what I mean; **(iii)** there's a sneaky trick that you'll need to find for part (e) which is related in some way to finding z^{-1} in complex numbers.]

(a) x^3, (b) x^4, (c) x^{-1}, (d) $\sin x$, (e) $x^{1/2}$.

Q2. Find the derivatives of the following functions with respect to x:

(a) $4\sin x + 2x$, (b) $4e^{2x} + 5x^{-1}$, (c) $(bx)^{-1}$, (d) $-4 - 5x^{-2}$, (e) e^{3x-4}, (f) $\ln|2x^3|$, (g) $|x|$, (h) $\sin|x|$.

Q3. Find the derivatives of the following functions with respect to t:

(a) $t\sin t$, (b) $t^{-2}e^{3t}$, (c) $t\ln t - t$, (d) $te^{-t}\cos 2t$, (e) $\sin 2t \sinh 3t$, (f) $|t|\sin|t|$.

Q4. Differentiate the following problems with respect to t:

(a) e^{t^2}, (b) $\sqrt{1+t^2}$, (c) $(1+\sqrt{t})^2$, (d) $\sin[\sin(\sin t)]$, (e) $\tan(t^{1/2})$, (f) $e^{-\sin t^2}$, (g) $(\sin t)^{1/2}$, (h) $|t|^b$.

Q5. Using the quotient rule, differentiate the following with respect to x:

(a) $\tan(ax)$, (b) $\tanh(ax)$, (c) $\operatorname{cosec}(ax)$, (d) $e^x/(1+x)$, (e) e^{3x}/x^2, (f) $x/(1+x^2)$.

Use $\tan = \sin/\cos$, $\tanh = \sinh/\cosh$ and $\operatorname{cosec} = 1/\sin$.

Q6. Find an expression for dy/dx in the following cases:

(a) $y^2 + y = x$, (b) $\sin(xy) = x$, (c) $\ln|y| = y - \cos x$.

Q7. Find the derivatives with respect to x of the following functions. You will need to use more than one of the above rules in some cases. Part (j) is rather lengthy.

(a) $\sin^{-1}(ax+b)$ (hint: let $y = \sin^{-1}(ax+b)$, find x in terms of y and then differentiate),

(b) $\sin^{-1}(\sin 2x)$, (c) $e^{x\sin x}$, (d) $(\sin x)e^{x^2}$,

(e) 2^x (hint: first show that $2^x = e^{x\ln 2}$), (f) x^x, (g) $\log_{10}|x|$,

(h) $\sinh^{-1}(ax+b)$, [N.b. this is an inverse sinh, not a reciprocal]

(i) $xe^{(x/\sqrt{1+x^2})}$,

(j) $\sin(x^2)e^{x\sin x}/(1+x^2)$.

Q8. **[Lengthy and challenging]** Find the first, second and third derivatives of $x^n e^{ax}$, where we can assume that $n > 3$. Can you write down a compact expression for the mth derivative of this function?

Q9. **[Challenging]** This one is the strangest one.... By considering a simple sketch it is easy to be convinced of the fact that $dy/dx = 1/(dx/dy)$. Use this result and the chain rule to find the appropriate formula for d^2y/dx^2 in terms of d^2x/dy^2. Check that your final formula is correct by applying it to $y = \ln x$ (for $x > 0$) and to $y = x^2$.

Q10. **To be answered after viewing video 3.** Find the critical points of the following functions. Which are maxima and which are minima? Find the values of the functions at these points and sketch the functions.

(a) $f(x) = 2x^2 - 3$, (b) $g(t) = t^2(t^2 - 1)$, (c) $h(y) = ye^{-y}$, (d) $F(x) = x^2 \sin^2 x$,

(e) $G(x) = x^4 + 2x^3 - 2x - 1$.

Q11. **To be answered after viewing video 3.** Find and classify any critical points of the function, $y = 10x^6 - 36x^5 + 45x^4 - 20x^3 + 1$.

Q12. **To be answered after viewing video 3.** Find and classify any critical points of the function, $y = e^x/(x^2 + 1)$. [Hint, this becomes very very messy after the first derivative. Therefore I would suggest that one should set $y' = (x - 1)^2 f(x)$ at that point before obtaining any higher derivatives.]