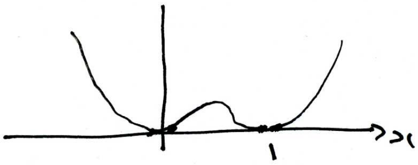
UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

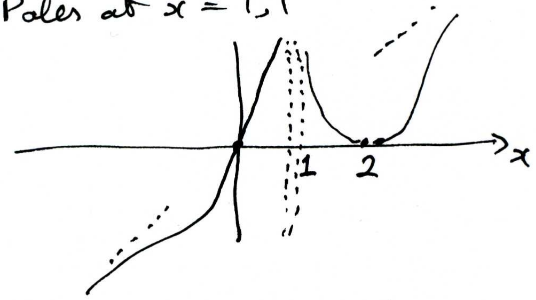
Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date Jan 2020
Unit Title: Mathematics 1		Unit Code: ME10304
Year: 2019/20	Question Number: 1	Page 1 of 1

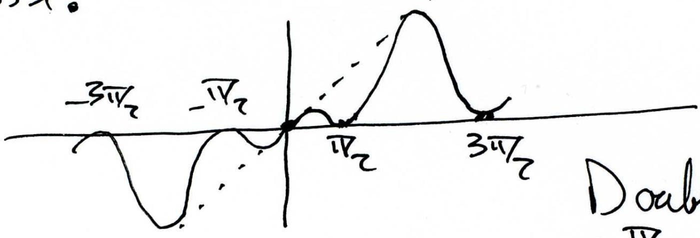
Part		Mark
1.	(a) $x^4-2x^3+x^2 = x^2(x-1)^2$. Double zeros at $x=0,1$.	2



(b) $\dfrac{x(x-2)^2}{(x-1)^2}$ Zeros at $x=0,2,2$ Poles at $x=1,1$ fct $\sim x$ when $|x|\gg 1$ — **2**



(c) $x\cos^2 x$. Use $0 \le \cos^2 x \le 1$. Hence — **3**



Double zeros at $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

(d) $\frac{1}{2}(1+\tanh x)$ — **3**

Alternatives are: $\frac{1}{\pi}\tan^{-1}x + \frac{1}{2}$,

$$\frac{1}{2}\left[\frac{x}{1+|x|}+1\right],$$

and

$$\frac{1}{2}\left[\frac{e^x-e^{-x}}{e^x+e^{-x}+1}+1\right].$$

Total	10

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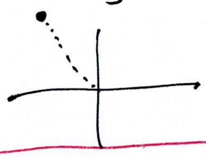
(a)	$\dfrac{2+j}{1-2j} = \dfrac{(2+j)(1+2j)}{(1-2j)(1+2j)} = \dfrac{5j}{5} = j.$ Conjugate is $-j.$	2
(b)	$2e^{j\pi/6} = 2\left[\cos\pi/6 + j\sin\pi/6\right] = \sqrt{3}+j$	2
(c)	Let $z = -7+24j = 25e^{j(\theta+2\pi n)}$, $n=0,1,2,3$ and $\theta = \tan^{-1}\dfrac{24}{-7} = \pi - 1.287002 = 1.854590$  Hence $\boxed{z^{1/4} = \sqrt{5}\, e^{j(\theta+2\pi n)/4}, \quad n=0,1,2,3}$	3
(d)	$\cos 4\theta + j\sin 4\theta = (c+js)^4$ $= c^4 + 4jc^3s - 6c^2s^2 - 4jcs^3 + s^4$ $= c^4 - 6c^2(1-c^2) + (1-c^2)^2 + 4jsc(c^2-s^2)$ $= 8c^4 - 8c^2 + 1 \quad + \quad \cdots$ $\Rightarrow \boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}$ $\boxed{\sin 4\theta = 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)}$	3

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Part		Mark
a	(i) $e^{2t}(2t^{-2}-2t^{-3}) = \boxed{2t^{-3}e^{2t}(t-1)}$	1
	(ii) $\dfrac{t[3e^{3t}\cos t - e^{3t}\sin t] - e^{3t}\cos t}{t^2} = \boxed{e^{3t}\left[\dfrac{(3t-1)\cos t - t\sin t}{t^2}\right]}$	2
	(iii) $\boxed{-2t\cos(t^2)\, e^{-\sin t^2}}$	3
b	$y = 3t^5 - 5t^3 \Rightarrow y' = 15(t^4 - t^2)$	
	So $y' = 0$ when $t = 0, \pm 1$.	
	$y'' = 15[4t^3 - 2t]$	
	So $y''(0) = 0$ — inconclusive at present	
	$y''(1) = 30$ — minimum	
	$y''(-1) = -30$ — maximum.	
	$y''' = 15[12t^2 - 2]$	4
	So $y'''(0) = -30 \Rightarrow$ descending inflexion pt.	

	Total	10

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Part		Mark
(a)	$f = \frac{x}{x^2+y^2} \Rightarrow \boxed{f_y = \frac{-2xy}{(x^2+y^2)^2}}$	
	$\& \Rightarrow f_x = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$	2
(b)	$z = 3x^3 + xy^2 - 9x$	
	Crit. pts: $z_x = 9x^2+y^2-9$, $z_y = 2xy$.	
	$z_y = 0 \Rightarrow x=0$ or $y=0$.	
	When $x=0$, $z_x = 0 \Rightarrow y = \pm 3$; When $y=0$, $z_x=0 \Rightarrow x = \pm 1$ — Four critical pts.	
	Classification: $z_{xx} = 18x$, $z_{yy} = 2x$, $z_{xy} = 2y$	

x	y	z_{xx}	z_{yy}	z_{xy}	H	conclusion
0	3	0	0	6	−36	Saddle
0	−3	0	0	−6	−36	Saddle
1	0	18	2	0	36	Min
−1	0	−18	−2	0	36	Max

$(H = z_{xx}z_{yy} - z_{xy}^2)$

Mark: 8

Total: 10

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Part		Mark
(a)	$\displaystyle\int_0^\pi t^2\cos t\,dt = \cancel{[t^2][\sin t]_0^\pi} - [2t][-\cos t]_0^\pi + \cancel{[2][-\sin t]_0^\pi}$ $= 2t\cos t\Big\vert_0^\pi = \boxed{-2\pi.}$	2
(b)	Let $x = \sin\theta$ in $I = \displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$ $\Bigg\Vert$ $dx = \cos\theta\,d\theta$; $x=0 \Rightarrow \theta=0$; $x=1 \Rightarrow \theta=\pi/2$ So $I = \displaystyle\int_0^{\pi/2} \frac{\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}} = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$ $= \boxed{\pi/4}$	3
(c)	Let $\dfrac{t+2}{t^3-t} = \dfrac{A}{t} + \dfrac{B}{t+1} + \dfrac{C}{t-1} \Longrightarrow t+2 = A(t^2-1) + Bt(t-1) + Ct(t+1)$. $t=0 \Rightarrow A=-2$ $t=1 \Rightarrow 2C=3$ or $C=3/2$ $t=-1 \Rightarrow 2B=1$ or $B=1/2$ Hence $\displaystyle\int \frac{t+2}{t^3-t}\,dt = \int\left[\frac{1/2}{t+1} + \frac{3/2}{t-1} - \frac{2}{t}\right]dt$ $= \frac{1}{2}\ln\vert t+1\vert + \frac{3}{2}\ln\vert t-1\vert - 2\ln\vert t\vert + c$ $= \boxed{\ln\left(\dfrac{\vert t+1\vert^{1/2}\vert t-1\vert^{3/2}}{t^2}\right) + c}$	5

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Part		Mark
(a)	$h = (x+2y)^2$	
	$V = \int_0^1\int_0^2 (x+2y)^2\,dx\,dy$	
	$= \int_0^1 \left[\frac{(x+2y)^3}{3} \right]_{x=0}^{x=2} dy = \frac{1}{3}\int_0^1 \left[(2+2y)^3 - (2y)^3\right] dy$	
	$= \frac{8}{3}\int_0^1 \left[(1+y)^3 - y^3\right] dy = \frac{8}{3}\int_0^1 \left[1+3y+3y^2\right] dy$	
	$= \frac{8}{3}\left[y + \frac{3y^2}{2} + y^3\right]_0^1 = \frac{8}{3}\left[\frac{7}{2}\right] = \boxed{\frac{28}{3}}$	3
(b)	$V = \pi\int_0^1 x^6\,dx = \boxed{\frac{\pi}{7}}$	
	$S = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$	
	Let $u = 1+9x^4 \Rightarrow du = 36x^3\,dx$	
	$x=0 \Rightarrow u=1$ $x=1 \Rightarrow u=10$	
	$\therefore\ S = 2\pi\int_1^{10} \frac{1}{36}u^{1/2}\,du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$	
	$= \boxed{\frac{\pi}{27}\left[10^{3/2}-1\right]} = 3.56312$ (6S.F.)	7

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Part		Mark
(a)		

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-2}$	1
1	$-2(1+x)^{-3}$	-2
2	$3!\,(1+x)^{-4}$	$3!$
3	$-4!\,(1+x)^{-5}$	$-4!$
4	$+5!\,(1+x)^{-6}$	$5!$

Hence $(1+x)^{-2} = 1 - 2x + \frac{3!\,x^2}{2!} - \frac{4!\,x^3}{3!} + \dots$

$= 1 - 2x + 3x^2 - 4x^3 + \dots$

$= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1) x^n}$ *See below. (3)

(b) $y = \ln(1+x) \Rightarrow y' = \frac{1}{1+x} = 1 - x + \frac{(-1)(-2)x^2}{2} - \frac{(-1)(-2)(-3)x^3}{3!} \dots$

$= 1 - x + x^2 - x^3 - \dots$

Integrate: $y = c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$y(0) = 0 \Rightarrow c = 0$. Hence $\boxed{y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$. (3)

(c) $u_n = \frac{(-1)^n x^{2n}}{2^n}$. So $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}\, 2^n}{2^{n+1} (-1)^n x^{2n}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{2} \right|$.

This must be less than 1. Hence $\boxed{|x| < \underline{\sqrt{2}}.}$ (2)

Radius of convergence

(d) $\lim_{x\to 0} \frac{x - \sin x}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{1-\cos x}{3x^2} \overset{H}{=} \lim_{x\to 0} \frac{\sin x}{6x}$

$(0/0)$ $\quad$ $(0/0)$ $\quad$ $(0/0)$

$\overset{H}{=} \lim_{x\to 0} \frac{\cos x}{6} = \boxed{1/6}.$ (2)

(a) Hence $(1-x^2)^{-2} = \sum_{n=0}^{\infty} (n+1) x^{2n}$

Total	10

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Part		Mark
8	(a) $\boxed{\begin{pmatrix}0\\0\\0\end{pmatrix}}$	1
	(b) $\underline{a}.\underline{b} = 8-1-7 = \boxed{0}$	1
	(c) Given (b), angle $= \boxed{90^\circ}$ or $\pi/2$ rad.	1
	(d) $\underline{b}.\underline{c} = 6+0-3 = \boxed{3}$	1
	(e) $\lvert\underline{b}\rvert = \boxed{\sqrt{6}}$	1
	(f) $\hat{\underline{b}} = \boxed{\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}}$	1
	(g) $\underline{a}\times\underline{b} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 4 & 1 & 7\\ 2 & -1 & -1\end{vmatrix}\begin{matrix} \\ 4 & 1\\ 2 & -1\end{matrix}$ $= \boxed{\begin{pmatrix}6\\18\\-6\end{pmatrix}}$	2
	(h) $\underline{b}.\underline{c} = \sqrt{6}\times\sqrt{18}\cos\theta = 6\sqrt{3}\cos\theta$ part (e) gives $\underline{b}.\underline{c} = 3$ $\Rightarrow 6\sqrt{3}\cos\theta = 3 \Rightarrow \cos\theta = \sqrt{3}/6$ $\Rightarrow \theta = \boxed{73.2213^\circ \text{ or } 1.277954 \text{ rad.}}$	2
	Total	10

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Part		Mark
(a)	$\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) = \begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$	
	Nearest approach to origin when $\underline{r}.(\underline{b}-\underline{a}) = 0$	3
	$\Rightarrow -3+3\lambda = 0 \Rightarrow \lambda = 1 \Rightarrow \underline{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \Rightarrow \boxed{\lvert\underline{r}\rvert = \sqrt{2}}$	
(b)	$\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) + \sigma(\underline{c}-\underline{a})$	
	$= \boxed{\begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\-1\\-1\end{pmatrix} + \sigma\begin{pmatrix}1\\-2\\-2\end{pmatrix}}$	3
(c)	$\underline{p} = (\underline{b}-\underline{a}) \times (\underline{c}-\underline{a}) = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 1 & -2 & -2 \end{vmatrix}\begin{matrix} -1 & -1 \\ 1 & -2 \end{matrix}$	
	$= \begin{pmatrix}0\\-3\\3\end{pmatrix}$	
	But $\lvert\underline{p}\rvert = \sqrt{18} = 3\sqrt{2}$	
	$\Rightarrow \hat{\underline{p}} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}}$	2
	$\underline{r}\cdot\hat{\underline{p}} = \underline{a}\cdot\hat{\underline{p}} + \underbrace{\cancel{\lambda(\underline{b}-\underline{a})\cdot\hat{\underline{p}}} + \cancel{\sigma(\underline{c}-\underline{a})\cdot\hat{\underline{p}}}}_{\text{by definition of } \hat{\underline{p}}}$	
	$\Rightarrow \frac{1}{\sqrt{2}}(-y+z) = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\end{pmatrix} = -\frac{1}{\sqrt{2}}.$	
	Hence the closest approach is $\boxed{\frac{1}{\sqrt{2}}}$	2
	Total	10

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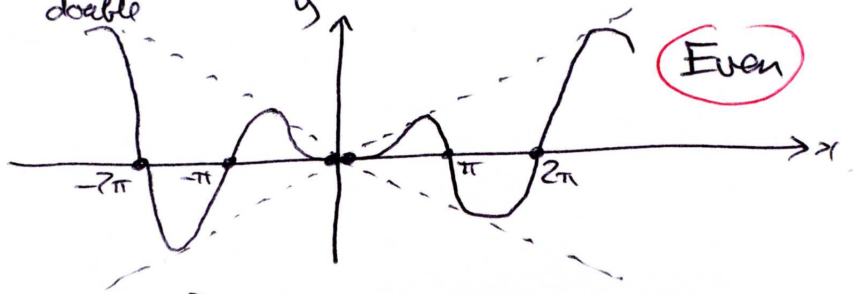
Part		Mark

(a) Zeros at $x = \underbrace{0,0}_{\text{double}}, \pm\pi, \pm 2\pi, \dots$



Even — 1

(b)

$$y' = \sin x + x\cos x$$
$$y'' = 2\cos x - x\sin x$$
$$y''' = -3\sin x - x\cos x$$
$$y^{iv} = -4\cos x + x\sin x$$

hence $\boxed{y^{(10)} = 10\cos x - x\sin x}$ — 3

(c) $I = \int_0^{2\pi} x\sin x\,dx = [x][-\cos x]_0^{2\pi} - [1][-\sin x]_0^{2\pi}$ — 2

$= \boxed{-2\pi}$

$\text{Mean} = \frac{1}{2\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{2\pi}I = \boxed{-1}$

(d) $V = \pi\int_0^{2\pi} x^2\sin^2 x\,dx = \frac{\pi}{2}\int_0^{2\pi} x^2(1-\cos 2x)\,dx$

$= \frac{\pi}{2}\left[\frac{x^3}{3}\right]_0^{2\pi} - \frac{\pi}{2}\left[\cancel{[x^2]\left[\frac{\sin 2x}{2}\right]} - [2x]\left[-\frac{\cos 2x}{4}\right] + \cancel{[2]\left[-\frac{\sin 2x}{8}\right]}\right]_0^{2\pi}$

$= \frac{4\pi^4}{3} - \frac{\pi}{2}\left[\frac{x}{2}\cos 2x\right]_0^{2\pi} = \boxed{\frac{4\pi^4}{3} - \frac{\pi^2}{2}} = 124.944$

$\text{RMS}^2 = \frac{1}{2\pi}\int_0^{2\pi} x^2\sin^2 x\,dx = \frac{1}{2\pi^2}V = \frac{2\pi^2}{3} - \frac{1}{4}$ — 4

$\Rightarrow \text{RMS} = \boxed{\left(\frac{2\pi^2}{3} - \frac{1}{4}\right)^{1/2}}$

$= 2.51590$

Total	10