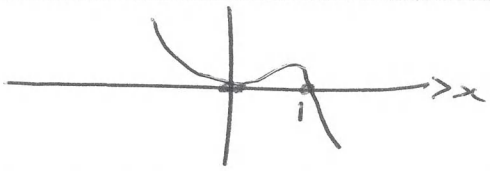
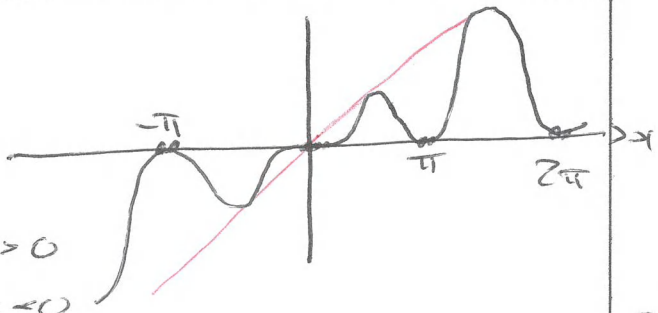
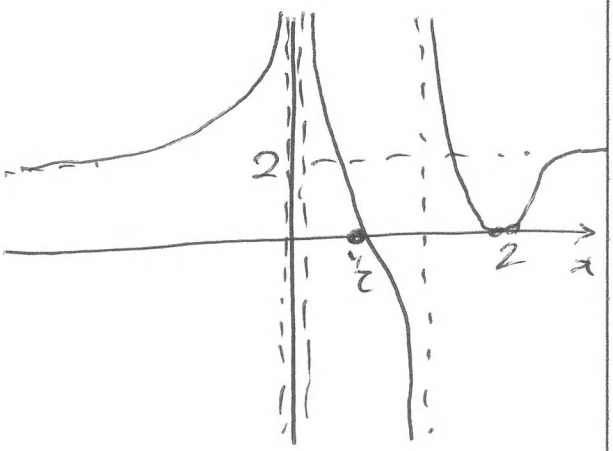
UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 1	Page 1 of 1	

Part		Mark
(a)	$y = x^2 - x^3 = x^2(1-x)$ Zeros @ $x = 0, 0, 1$ $y \to -\infty$ as $x \to +\infty$ 	2
(b)	$y = \sinh x + \cosh x = e^x$ 	2
(c)	$y = x \sin^2 x$ Zeros at $x = 0, 0, 0$ + double zeros at $x = \pm n\pi$. Also $0 \le y \le x$ when $x > 0$ & $x \le y \le 0$ when $x < 0$ 	3
(d)	$y = \dfrac{(2x-1)(x-2)^2}{x^2(x-1)}$ Zeros at $x = 2, 2, \frac{1}{2}$ Poles at $x = 0, 0, 1$ As $x \to \pm\infty$, $y \to 2$ 	5
		-
	Total 10	

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 2	Page 1 of 1	

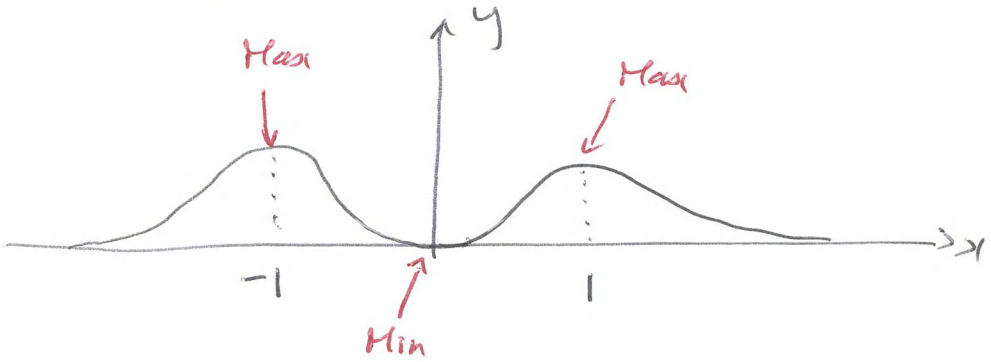
Part		Mark
(a)	$z = \frac{(2+j)^2}{1+7j} = \frac{(3+4j)(1-7j)}{(1+7j)(1-7j)} = \frac{11-2j}{5}$ * $z = \sqrt{5}\, e^{j\theta}$ where $\tan\theta = -\frac{2}{11}$ $\Rightarrow \theta = -0.179824$	3
(b)	$z = -8 \pm 15j = 17 e^{j(\theta + 2n\pi)}$ where $\theta = \tan^{-1}\left(-\frac{15}{8}\right)$ $\Rightarrow \theta = -1.080839 + \pi = 2.060754$ $n = 0, 1, 2$ $\Rightarrow z^{1/3} = 17^{1/3} e^{j(\theta + 2n\pi)/3}$	2
(c)	Use $c5 + j s5 = (c + js)^5$ $= c^5 + 5jc^4s - 10c^3s^2 - 10jc^2s^3 + 5cs^4 + js^5$ $\Rightarrow s5 = s[5c^4 - 10c^2s^2 + s^5] = s[5(1-s^2)^2 - 10s^2(1-s^2) + s^4]$ $= s[5 - 20s^2 + 16s^4]$ If $\theta = \pi/5$, then $\sin 5\theta = \sin\pi = 0 \Rightarrow 16s^4 - 20s^2 + 5 = 0$ $\Rightarrow s^2 = \frac{20 \pm [400 - 4\times5\times16]^{1/2}}{32} = \frac{5 \pm \sqrt{5}}{8}$ Need -ve sign (+ve is 72°) $\Rightarrow s = \sqrt{\frac{5 - \sqrt{5}}{8}}$.	5
	Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 3	Page 1 of 1	

Part		Mark
(a)	(i) $y' = \cos t \sec^2 t - 2\sin t \sin^2 t$	1
	(ii) $y' = -\frac{1}{\sin^2 t}$ or $-\operatorname{cosec}^2 t$	2
	(iii) $y' = \frac{1 + 2te^{-t^2}}{2\sqrt{t} + e^{-t^2}}$	3
(b)	$y = t^2 e^{-t^2} \Rightarrow y' = e^{-t^2}[2t - 2t^3]$	
	So $y' = 0$ when $t = 0, \pm 1$.	
	But $y'' = e^{-t^2}[2 - 6t^2 - 4t^2 + 4t^4]$	
	$= e^{-t^2}[2 - 10t^2 + 4t^4]$	
	At $t = 0$ $y'' = 2$ $\Rightarrow$ min.	
	$t = \pm 1$ $y'' = -4e^{-1}$ $\Rightarrow$ max.	



Total	

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022
Unit Title: Mathematics 1		Unit Code: ME10304
Year: 2021/22	Question Number: 4	Page 1 of 1

Part		Mark
(a)	$f = \frac{y}{x^2+y^2} \Rightarrow f_x = \frac{-2xy}{(x^2+y^2)^2}$ And $f_y = \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$	4
(b)	$z = y(y - x^2 + 3) = y^2 - yx^2 + 3y$ But $z_x = -2xy$ and $z_y = 2y - x^2 + 3$ If $x=0$, then $2y+3=0 \Rightarrow y = -3/2$ If $y=0$, then $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ Three critical points So $z_{xx} = -2y \quad z_{yy} = 2 \quad z_{xy} = -2x$	

x	y	z_{xx}	z_{yy}	z_{xy}	H	
$\sqrt{3}$	0	0	0	$-2\sqrt{3}$	-12	Saddle
$-\sqrt{3}$	0	0	0	$-2\sqrt{3}$	-12	Saddle.
0	$-3/2$	3	2	0	6	Min

6

Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 5	Page 1 of 1	

Part		Mark
(i)	$\int_0^\infty t^3 e^{-7t}\,dt = \left[t^3\left[\frac{e^{-7t}}{-7}\right]_0^\infty - \left[3t^2\left[\frac{e^{-7t}}{49}\right]_0^\infty + \left[6t\left[\frac{e^{-7t}}{-343}\right]_0^\infty - \left[6\left[\frac{e^{-7t}}{2401}\right]_0^\infty\right.\right.\right.\right.$ $= \frac{6}{16} = 3/8$	(2)
(ii)	$I = \int_0^1 \frac{x\,dx}{(1-x^2)^{1/2}}$ let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$ $x=0 \Rightarrow \theta=0$ $x=1 \Rightarrow \theta=\pi/2$ $\Rightarrow I = \int_0^{\pi/2} \frac{\sin\theta}{(1-\sin^2\theta)^{1/2}}\cos\theta\,d\theta = \int_0^{\pi/2} \sin\theta\,d\theta = 1$	(3)
(iii)	$I = \int \frac{s^2+3}{s^3-s}\,ds$. Now $s^3-s = s(s-1)(s+1)$ so let $\frac{s^2+3}{s^3-s} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$ $\Rightarrow s^2+3 = A(s^2-1) + Bs(s+1) + Cs(s-1)$. $s=0 \Rightarrow 3 = -A$ $\Rightarrow A = -3$ $s=1 \Rightarrow 4 = 2B$ $B = C = 2$. $s=-1 \Rightarrow 4 = 2C$ So $I = \int \left[\frac{2}{s-1} + \frac{2}{s+1} - \frac{3}{s}\right] ds$ $= 2\ln\lvert s-1\rvert + 2\ln\lvert s+1\rvert - 3\ln s + c$ $= \ln\left\lvert\frac{(s^2-1)^2}{s^3}\right\rvert + c$	(5)
	Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 6	Page 1 of 1	

Part		Mark

(a) $$V = \int_0^{\pi/4}\int_0^{2} r^2\cos^2\theta \, r\,dr\,d\theta = \int_0^{\pi/4} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \times \int_0^{2} r^3\,dr$$

$$= \frac{\pi}{4} \times 4 = \pi$$

(b) $y = x^3 \quad 0 \le x \le 1$

$$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 x^6\,dx = \frac{\pi}{7}$$

$$S = 2\pi\int_0^1 y\sqrt{1+y'y'}\,dx = 2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$$

Let $v = 1+9x^4 \Rightarrow dv = 36x^3\,dx$

$x = 0 \Rightarrow v = 1$

$x = 1 \Rightarrow v = 10$

Hence

$$S = 2\pi\int_1^{10} \frac{1}{36}\sqrt{v}\,dv = \frac{\pi}{27}\left(10^{3/2} - 1\right)$$

$$= 3.563127$$

Total	

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 7	Page 1 of 1	

Part		Mark
(a)	n | $\sinh^{(n)} x$ | $\sinh^{(n)} 0$ 0 | $\sinh x$ | 0 1 | $\cosh x$ | 1 2 | $\sinh x$ | 0 3 | $\cosh x$ | 1 4 | $\sinh x$ | 0 $\Rightarrow \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	(2)
(b)	$(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \ldots$ $= 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$	(2)
(c)	$u_n = \frac{n!\,n!}{(2n)!} x^{2n} \longrightarrow \frac{u_{n+1}}{u_n} = \frac{(n+1)!(n+1)!\,x^{2n+2}\,(2n)!}{n!\;n!\;x^{2n}\,(2n+2)!}$ $= \frac{(n+1)^2}{(2n+2)(2n+1)} x^2$ $\longrightarrow \frac{x^2}{4}$ as $n \to \infty$ Need $\left\|\frac{x^2}{4}\right\| < 1$ for convergence $\Rightarrow |x| < 2$. Radius of convergence	(3)
(d)	$\lim_{x\to 0} \frac{\cosh 2x - \cos x}{x^2} = \lim_{x\to 0} \frac{2\sinh 2x + \sin x}{2x}$ $= \lim_{x\to 0} \frac{4\cosh 2x + \cos x}{2} = \frac{5}{2}$	(3)
	Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 8	Page 1 of 1	

Part		Mark
-	(a) $\begin{pmatrix} -2 \\ -5 \\ 9 \end{pmatrix}$ (b) 9 (c) 3 (d) 9 (e) $\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ (f) 1,230959 rad or 70,5288° (g) 90° (h) $\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 1 & -4 & 8 \end{vmatrix} = \begin{pmatrix} 24 \\ -6 \\ -6 \end{pmatrix}$	-
	Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

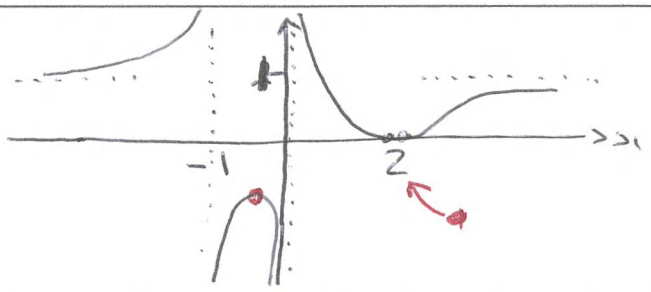
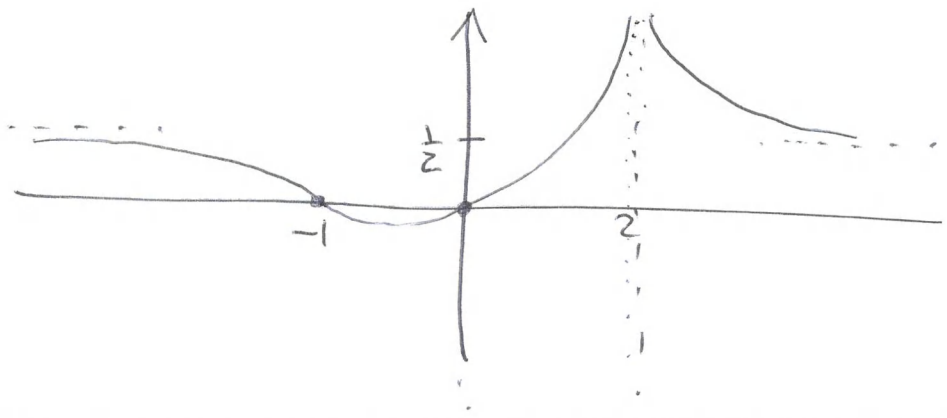
Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 9	Page 1 of 1	

Part		Mark
(a)	$\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\-1\\0\end{pmatrix}$ Need $\underline{r}\cdot(\underline{b}-\underline{a}) = 0 \Rightarrow \lambda = 0 \Rightarrow \underline{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \lvert\underline{r}\rvert = \sqrt{3}$	2
(b)	$\underline{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix} + \sigma\begin{pmatrix}2\\1\\2\end{pmatrix}$	2
(c)	$\underline{p} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\1 & 0 & -1\\2 & 1 & 2\end{vmatrix} = \begin{pmatrix}1\\-4\\1\end{pmatrix} \Rightarrow \hat{p} = \frac{1}{\sqrt{18}}\begin{pmatrix}1\\-4\\1\end{pmatrix}$ So $\underline{r}\cdot\hat{p} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-4\\1\end{pmatrix}\frac{1}{\sqrt{18}} = \frac{-2}{\sqrt{18}} = -\sqrt{2/9}$	3
(d)	Here $(x,y,z)\cdot(1,-4,1)\frac{1}{\sqrt{18}} = -\sqrt{\frac{4}{18}}$ $\Rightarrow \boxed{x - 4y + z = -2}$ Intersection with $z=0$: $x - 4y = -2$ or $y = \frac{x}{4} + \frac{1}{2}$	3
	Total	10

UNIVERSITY OF BATH
DEPARTMENT OF MECHANICAL ENGINEERING

Outline Solution to Examination Question

Examiner: Dr D A S Rees		Date January 2022	
Unit Title: Mathematics 1		Unit Code: ME10304	
Year: 2021/22	Question Number: 10	Page 1 of 1	

Part		Mark
(a)	$y = \frac{(x-2)^2}{x^2+x}$ Zeros at $x = 2, 2$ Poles at $x = 0, -1$ $y \to 1$ at $x \to \pm\infty$ 	2
(b)	$y' = \frac{(x^2+x)(2x-4) - (x^2-4x+4)(2x+1)}{(x^2+x)^2} = \frac{5x^2-8x-4}{(\quad)^2}$ Need $5x^2-8x-4=0 \Rightarrow x = 2, -0.4$ See red dots.	2
(c)	$\frac{x^2-4x+4}{x^2+x} = \frac{x^2+x+(-5x+4)}{x^2+x}$ $= 1 + \frac{(-5x+4)}{x(x+1)} = 1 + \frac{4}{x} - \frac{9}{x+1}$ Using Partial Fractions. $\Rightarrow \int \frac{(x-2)^2}{(x^2+x)}\,dx = x + 4\ln\lvert x\rvert - 9\ln\lvert x+1\rvert + C$ $= x + \ln\left\lvert \frac{x^4}{(x+1)^9}\right\rvert + C$ $\Rightarrow \int_1^3 \frac{(x-2)^2}{(x^2+x)}\,dx = 2 + \ln\frac{81}{512} = 0.156175$	4
(d)		2
	Total	10