UNIVERSITY OF BATH

DEPARTMENT OF MECHANICAL ENGINEERING

ME10304

MATHEMATICS 1

Tuesday 17th January 2023
09.30-11.30
2 hours

The examination consists of TEN questions each with 10 marks available.
Attempt all questions.

The marks shown against each **part** of a question are for guidance only.

Candidates may only use University-supplied calculators.

Tables of basic formulae are provided.

Students are allowed to use a pre-prepared two-sided A4 page of crib notes.

PLEASE FILL IN THE DETAILS ON THE FRONT OF YOUR ANSWER BOOK AND SIGN IN THE SECTION ON THE RIGHT OF YOUR ANSWER BOOK, PEEL AWAY ADHESIVE STRIP AND SEAL.

TAKE CARE TO ENTER THE CORRECT CANDIDATE NUMBER AS DETAILED ON YOUR DESK LABEL

DO NOT TURN OVER YOUR EXAM PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR

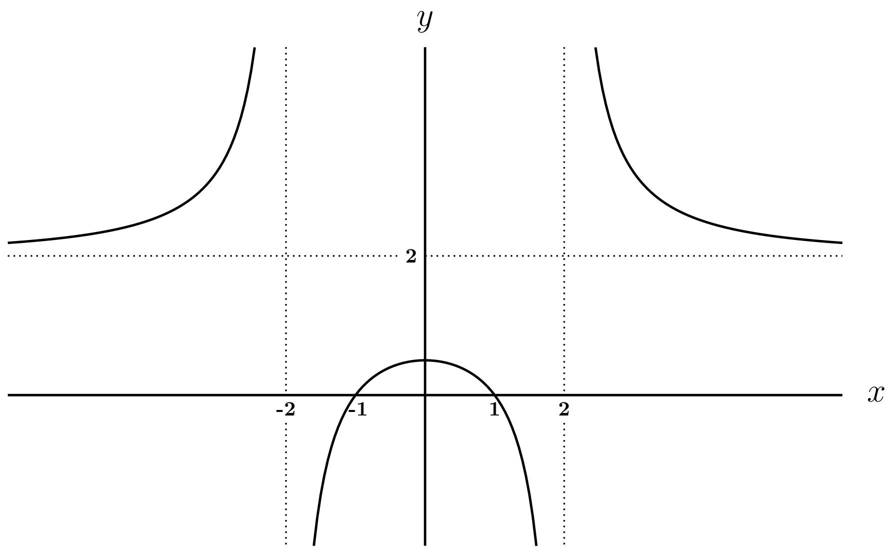
1. (a) Sketch the behaviour of y in terms of x for the following three cases, making sure that all salient features are labelled suitably:

   (i) $y = xe^{-x^2}$, [2 marks]

   (ii) $y = \dfrac{\sin x}{1+x^2}$, [2 marks]

   (iii) $y^2 = x^2(1+x)$. [3 marks]

   (b) Write down a plausible expression for the function sketched below. [3 marks]



2. (a) Evaluate the complex number, $\dfrac{(1+j)(2+j)}{1+3j}$, and express it in both Cartesian form and complex exponential form. [4 marks]

   (b) Write $2e^{j\pi/6}$ in Cartesian form. [2 marks]

   (c) Find all the possible values of $(-5+12j)^{1/4}$ and sketch their location in the complex plane. [4 marks]

3. (a) Find the derivatives of the following functions with respect to t:

$$\text{(i)}\ \ \ln|3t|, \qquad \text{(ii)}\ \ \frac{t^2}{\cos t}, \qquad \text{(iii)}\ \ \sin\left(te^{-t^2}\right).$$

[1,2,3 marks]

   (b) Find all the critical points of the function, $y = 4t^3 - 3t^4 - 1$, and classify them. Hence sketch the function, labelling all the zeros and critical points clearly. [4 marks]

4. (a) Find both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when

$$f(x, y) = \frac{x + y}{x^2 + y^2}.$$

[3 marks]

(b) Find and classify all the critical points of the function,

$$z(x, y) = xy(y + x - 3).$$

[7 marks]

5. (a) Use the method of partial fractions to evaluate the definite integral,

$$\int_1^2 \frac{1}{x^3 + x^2}\, dx.$$

[5 marks]

(b) By first using the substitution, $x = e^{-y}$, find the value of the integral,

$$\int_0^1 x^2 \ln x \, dx.$$

[5 marks]

6. (a) Find the volume under the surface,

$$h(r, \theta) = r^2 \cos^2 \theta,$$

in the ranges, $0 \le r \le 2$ and $0 \le \theta \le \frac{1}{2}\pi$. [Here, r is the radial coordinate and θ is the angular coordinate.] [3 marks]

(b) Find both the surface area and the volume of the shape which is obtained by rotating the function, $y = x^3$, about the x-axis. Consider the following range of values of x: $0 \le x \le 1$. [7 marks]

7. (a) Find the Taylor's series representation of the function, $y = \sinh x$, about $x = 0$ (i.e. the Maclaurin series) and write it in summation form. [2 marks]

(b) Use the solution of Q7a to find,

$$\lim_{x \to 0} \frac{\sinh x - x}{x^3},$$

and confirm your result using l'Hôpital's rule. [4 marks]

(c) Find the binomial expansion of the function $y = (1 - x)^{-2}$ and express it in summation notation. [2 marks]

(d) Use d'Alembert's test to determine the radius of convergence of the series found in Q7c. [2 marks]

8. Given the vectors,

$$\underline{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \qquad \text{and} \qquad \underline{c} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix},$$

evaluate the following:

(a) $\underline{a} + \underline{b} - \underline{c}$ (b) $\underline{a} \cdot \underline{b}$ (c) $|\underline{a}|$

(d) $|\underline{b}|$ (e) $\hat{\underline{a}}$ (f) the angle between $\underline{a}$ and $\underline{b}$

(g) the angle between $\underline{b}$ and $\underline{c}$ (h) $\underline{a} \times \underline{b}$.

[Note that each part is allocated 1 mark apart from part (h) which is allocated 3 marks.] [10 marks]

9. The following are the position vectors of the points, A, B, and C, respectively,

$$\overline{OA} = \underline{a} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}, \qquad \overline{OB} = \underline{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \overline{OC} = \underline{c} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

(a) A line passes through points A and B; write down the equation of this line. How close does this line pass to the origin? [2 marks]

(b) A plane passes through A, B and C; find both $\underline{b} - \underline{a}$ and $\underline{c} - \underline{a}$ and hence write down the equation of that plane using two free parameters. [2 marks]

(c) Determine a unit vector, $\hat{\underline{p}}$, which is perpendicular to the plane. By taking the scalar product of the equation for the plane which was obtained in part (b) with $\hat{\underline{p}}$, determine how close the plane gets to the origin. [3 marks]

(d) Using the results of part (c), write down the equation of the plane in Cartesian form. Hence write down the equation of the line which forms the intersection of this plane with the plane, $z = 0$. [3 marks]

10. This question involves the evaluation of a definite integral of xe^{-x^2}. A numerical value is found via a power series expansion and this will be compared with the analytical solution .

(a) Use a Taylor's series approach to find a power series expansion for e^x about $x = 0$. Hence write down a power series representation for e^{-x^2}, and thence a series for the function, xe^{-x^2}. [4 marks]

(b) Use the first four terms in the above power series expansion for xe^{-x^2} to obtain a numerical approximation for

$$\int_0^{0.5} xe^{-x^2}\, dx;$$

please retain at least six decimal places in your solution. [3 marks]

(c) Use a suitable substitution to obtain the analytical value of the above integral. How accurate is the use of the Taylor's series method? [3 marks]