Department of Mechanical Engineering, University of Bath

Mathematics ME10304

Problem sheet 12 — Some example exam Q10 questions

Three years ago was the first time that I had introduced into the Mathematics exam a question 10 which was composed of a variety of techniques and ideas drawn from many different parts of the syllabus. Therefore the following questions are an indication of the type of question which I might possibly set. I have to say that my enthusiasm has overridden my sense of what consistutes a fair length for a question in some cases here (especially Q4), and although Q5 was on the the 2017/18 paper, some of the others are longer than you'll meet in January. The aim here is simply to give you an indication of how I might construct question 10 in the exam, and the last three years' papers show how much I intend to set this year.

1. The question concerns the function $y = x/\sqrt{1-x^2}$.

(a) Find the derivative of $y(x)$ and express it in as simple a form as possible.

(b) Sketch $y(x)$.

(c) Find the integral of $y(x)$ between $x = 0$ and $x = 1$.

(d) Find the volume of the surface of revolution of $y(x)$ about the x-axis between $x = 0$ and $x = \frac{1}{2}$.

(e) Find the volume of the surface of revolution of $y(x)$ about the y-axis between $y = 0$ and $y = 1$.

2. (a) Find a power series representation of $y = (1-x)^{-1/2}$ using either a Taylor's series or a Binomial series approach.

(b) Hence write down a power series representation of $(1-x^2)^{-1/2}$.

(c) Apply d'Alembert's test to the series obtained in part (a) to determine its radius of convergence.

(d) Given that,

$$\sin^{-1} y = \int_0^y \frac{dx}{(1-x^2)^{1/2}},$$

find a power series representation of $\sin^{-1} y$.

3. In this question we consider various aspects of the function,

$$y = \frac{(x-1)^2}{x^2+2x}.$$

(a) Sketch this function making sure that all salient features are described clearly.

(b) Find all of the critical points and add these to the sketch. (There is no need to categorise them.)

(c) Determine the definite integral of y between $x = 1$ and $x = 2$.

(d) Sketch the reciprocal of y.

4. The so-called sinc function is defined as $\operatorname{sinc} x = \dfrac{\sin x}{x}$.

(a) Use a Taylor/Maclaurin series approach to find a power series representation of $\sin x$ in terms of powers of x.

(b) Use the result of part (a) to determine $\lim_{x\to 0} \operatorname{sinc} x$.

(c) Confirm the result of part (b) by using l'Hôpital's rule to find $\lim_{x\to 0} \operatorname{sinc} x$.

(d) Sketch $\operatorname{sinc} x$ making sure that all salient features are described clearly.

(e) After writing the power series form for $\operatorname{sinc} x$, integrate it to find a power series representation for $\int_0^X \operatorname{sinc} x \, dx$. Write this series in summation form. Determine its radius of convergence using d'Alembert's test.

(f) Find the x-derivative of $\operatorname{sinc} x$ and hence show that all the critical points occur when $\tan x = x$.

(g) Find the second x-derivative of $\operatorname{sinc} x$, and hence show that the sinc function satisfies the differential equation,

$$xy'' + 2y' + xy = 0.$$

5. In this question we consider various aspects of the function,

$$y = \frac{x^2}{x^2 + 8x + 12}.$$

(a) Sketch this function making sure that all salient features are shown and described clearly.

(b) Find all of the critical points and add these to the sketch. (There is no need to categorise them.)

(c) Determine the definite integral of y between $x = 0$ and $x = 1$.

(d) Sketch the reciprocal of y.