Department of Mechanical Engineering, University of Bath

Mathematics 1 ME10304

Problem Sheet 10 — Vectors

I will use the row vector notation (a, b, c) to represent a three-dimensional vector, rather than $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (which is suitable for use with matrices) in order to make the typesetting look tidier and to save some space. An alternative notation is $(a, b, c)^T$ where the "T" superscript denotes Transpose. I have been lazy and have omitted the "T".

Q1. Given the vectors, $\underline{a} = (1, 0, 1)$, $\underline{b} = (2, 1, 2)$ and $\underline{c} = (0, 1, 1)$, evaluate the following:

(a) $\underline{a} + \underline{b}$ (b) $\underline{a} + \frac{1}{2}\underline{b} + 2\underline{c}$ (c) $\underline{b} - 2\underline{a}$ (d) $|\underline{a}|$ (e) $|\underline{b}|$

(f) $|\underline{a} - \underline{b}|$ (g) $\hat{\underline{a}}$ (h) $\hat{\underline{b}}$ (i) $\underline{a} \times \underline{b}$

(j) The angle between $\underline{a}$ and $\underline{b}$ (k) The angle between $\underline{b}$ and $\underline{c}$.

Q2. Use vectorial methods to find the internal angles of the triangle ABC where

(i) $$\overrightarrow{OA} = (1, 3), \qquad \overrightarrow{OB} = (4, 6), \qquad \overrightarrow{OC} = (-3, 7),$$

(ii) $$\overrightarrow{OA} = (-1, 3, 1) \qquad \overrightarrow{OB} = (2, 6, 2) \qquad \overrightarrow{OC} = (3, 3, 8).$$

[Hint: Draw a schematic of the triangle and the origin; you'll need to find vectors such as $\overrightarrow{AB}$.]

Q3. Find the unit vector which is perpendicular to both $(1, 4, 1)$ and $(2, 1, -1)$.

Q4. Find the value of α for which the following two vectors are perpendicular: $(1, 2, 7)$ and $(4, 5, \alpha)$. Determine the vector product of these vectors, and check that your answer is correct by taking the appropriate scalar products.

Q5. Prove that the four points with position vectors $(2, 1, 0)$, $(2, -2, -2)$, $(7, -3, -1)$ and $(12, 2, 4)$ are coplanar.

Q6. The vectors $(2, -4, -1)$, $(3, 2, -2)$ and $(5, -2, -3)$ correspond to the directions and lengths of the three sides of a triangle. Confirm that they are co-planar. What is the length of each side? Show that the triangle is right angled. There are three different cross products that may be found upon taking different pairs of the above vectors; find them and make a comment about the result.

Q7. My house has a south-facing aspect. From my front door I walk 1 mile south, followed by 1 mile east and then 1 mile north, whereupon I have returned to my house. A bear chooses that moment to walk past. What colour is the bear?

Q8. Lagrange's identity is

$$|\underline{a} \times \underline{b}|^2 + |\underline{a} \cdot \underline{b}|^2 = |\underline{a}|^2 |\underline{b}|^2.$$

Can you prove it? [Hint: this is ridiculously easy.]

Q9. This is a lengthy question which is of some interest but the content isn't examinable. In this we derive quite an interesting formula.

(a) The triangle OAB lies in the (x, y)–plane and has one vertex at the origin. Regard $\underline{k}$ as being the unit vector in the z-direction. The other two vertices are at the points with the position vectors, $\underline{a}$ and $\underline{b}$. Use one of the definitions of the vector cross product to show that the area of this triangle is given by

$$\mathcal{A} = \tfrac{1}{2}\,(\underline{a} \times \underline{b}) \cdot \underline{k}\,.$$

(b) One may now find the area of a regular pentagon with unit 'radius' by saying that it is 5 times the area of the triangle with vertices, $(0, 0)$, $(1, 0)$ and $(\cos \frac{2}{5}\pi, \sin \frac{2}{5}\pi)$. What is the area of a regular pentagon? What is the area of a regular dodecagon?

(c) Extend the result of (b) and find a formula for the area of a regular polygon with n sides. By setting $m = 1/n$, use l'Hôpital's rule on this formula to show that the area tends towards π as $n \to \infty$.

(d) It is now possible to find the area of irregular polygons by splitting them into triangles with the origin as one vertex and two neighbouring points as the other two vertices. Suppose we have a polygon with n vertices, where vertex i is located at (x_i, y_i). Using the formula given in part (a), show that the area of the polygon may be written as,

$$\mathcal{A} = \tfrac{1}{2}\sum_{i=1}^{n}\Big[x_i y_{i+1} - x_{i+1} y_i\Big],$$

where $(x_{n+1}, y_{n+1}) = (x_1, y_1)$, for notational convenience.

(e) Use the formula given in part (d) to find the areas corresponding to the following polygons:

(i) The triangle, $(1, 1)$, $(1, 3)$, $(-1, 2)$, $(1, 1)$;

(ii) The triangle, $(-1, 2)$, $(1, 3)$, $(1, 1)$, $(-1, 2)$;

(iii) The bowtie, $(1, 1)$, $(1, 2)$, $(-1, 1)$, $(-1, 2)$, $(1, 1)$.

Some aspects of the results will need some discussion.