

An extra example.

$$M = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & -1 \\ 3 & -1 & 6 \end{pmatrix}$$

$$0 = |M - \lambda I|$$

$$\begin{pmatrix} 3-\lambda & 0 & 3 \\ 0 & 3-\lambda & -1 \\ 3 & -1 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 0 &= \begin{vmatrix} 3-\lambda & 0 & 3 \\ 0 & 3-\lambda & -1 \\ 3 & -1 & 6-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ -1 & 6-\lambda \end{vmatrix} + 3 \begin{vmatrix} 0 & 3-\lambda \\ 3 & -1 \end{vmatrix} \\ &= (3-\lambda) [(3-\lambda)(6-\lambda) - 1 - 9] = (3-\lambda) [\lambda^2 - 9\lambda + 8] \\ &= (3-\lambda)(\lambda-1)(\lambda-8) \implies \boxed{\lambda = 1, 3, 8} \end{aligned}$$

$$\lambda = 1: \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Eq 1} &\Rightarrow 2x + 3z = 0 \Rightarrow x = -3A \Rightarrow z = A \\ \text{Eq 2} &\Rightarrow 2y - z = 0 \Rightarrow y = -A \end{aligned}$$

$$\underline{x} = A \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda = 3: \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Eq 1} &\Rightarrow z = 0 \\ \text{Eq 3} &\Rightarrow x = B \Rightarrow y = 3B \end{aligned}$$

$$\underline{x} = B \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\lambda = 8: \begin{pmatrix} -5 & 0 & 3 \\ 0 & -5 & -1 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Eq 1} &\Rightarrow x = 3C \Rightarrow z = 5C \\ \text{Eq 2} &\Rightarrow y = -C \end{aligned}$$

$$\underline{x} = C \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$