1.4 Separation of variables

This technique applies to 1st order ODEs which take the form,

$$\boxed{\frac{dy}{dt} = f(y)g(t),}$$

where f and g are given functions.

The general solution is obtained by **separating the variables,** i.e. by taking all y-dependent terms to one side and all t-dependent terms to the other, and then integrating. Thus the general solution follows in this manner:

$$\boxed{\begin{aligned} & \frac{dy}{dt} = f(y)g(t) \\ \Longrightarrow \quad & \frac{dy}{f(y)} = g(t)\,dt \\ \Longrightarrow \quad & \int \frac{dy}{f(y)} = \int g(t)\,dt. \end{aligned}}$$

So the method of solving a **variables-separable** equation reduces down to the finding of two integrals. The final solution is them obtained when an initial condition is applied.

Example 1.1: Solve the ODE, $\boxed{\dfrac{dy}{dt} = 2\sqrt{y}\,\cos t}$ subject to $\boxed{y(0) = 0}$.

Clearly this is a variables-separable ODE and therefore we may proceed as above:

$$\frac{dy}{dt} = 2\sqrt{y}\,\cos t$$

$$\Longrightarrow \quad \frac{dy}{\sqrt{y}} = 2\cos t\,dt \qquad \text{Variables separated!}$$

$$\Longrightarrow \quad \int \frac{dy}{\sqrt{y}} = \int 2\cos t\,dt$$

$$\Longrightarrow \quad 2\sqrt{y} = 2\sin t + c \qquad c \text{ is arbitrary}$$

$$\Longrightarrow \quad \sqrt{y} = \sin t + \tfrac{1}{2}c$$

$$\Longrightarrow \quad y = (\sin t + \tfrac{1}{2}c)^2 \qquad \text{Now apply the Initial Condition } \ldots$$

$$\Longrightarrow \quad y = \sin^2 t \qquad y(0) = 0 \;\Rightarrow\; c = 0$$

Note: This is the typical way that a separation-of-variables problem proceeds. The final solution is as difficult to obtain as the integrals are which make it up.

In the next two examples we will consider slightly different cases where the respective presence of a square root and of a logarithm will require some careful treatment.

Example 1.2: Solve the ODE, $\boxed{y' = 3t^2/y}$ subject to $\boxed{y(1) = 2}$. This may be undertaken in two slightly different ways.

- We have

$$\boxed{\frac{dy}{dt} = \frac{3t^2}{y} \quad \Longrightarrow \quad \int y\,dy = \int 3t^2\,dt \quad \Longrightarrow \quad \tfrac{1}{2}y^2 = t^3 + c \quad \Longrightarrow \quad y = \pm\sqrt{2t^3 + 2c}.}$$

  Application of the initial condition shows that $2 = \pm\sqrt{2 + 2c}$. The only way that this can be solved is for $c = 1$ to be chosen *and* the positive sign to be taken. The final solution is

$$y = \sqrt{2t^3 + 2}.$$

- The alternative route is as follows.

$$\boxed{\frac{dy}{dt} = \frac{3t^2}{y} \quad \Longrightarrow \quad \int y\,dy = \int 3t^2\,dt \quad \Longrightarrow \quad \tfrac{1}{2}y^2 = t^3 + c.}$$

  Now apply the initial condition, $y(1) = 2$ to obtain $c = 1$. Taking the square root means that $y = \pm\sqrt{2t^3 + 2}$, and we need to appeal a second time to the initial condition in order to confirm that we need the positive square root.

Example 1.3: Solve the ODE, $\boxed{y' = 2ty}$, subject to $\boxed{y(0) = -2}$.

We get $$\boxed{\int \frac{dy}{y} = \int 2t\,dt \qquad \Longrightarrow \qquad \ln|y| = t^2 + c.}$$

Application of the IC gives $c = \ln 2$. Hence $\boxed{\ln|y| = t^2 + \ln 2.}$

$$\boxed{e^{\text{LHS}} = e^{\text{RHS}} \quad \Rightarrow \quad |y| = e^{t^2 + \ln 2} \quad \Rightarrow \quad |y| = 2e^{t^2} \quad \Rightarrow \quad y = \pm 2e^{t^2}.}$$

Application of the IC a second time confirms the negative form. Hence the solution is $y = -2e^{t^2}$.

However, if one wishes to obtain y *before* applying the IC, then we get,

$$\boxed{\int \frac{dy}{y} = \int 2t\,dt \quad \Rightarrow \quad \ln|y| = t^2 + c \quad \Rightarrow \quad |y| = e^{t^2 + c} = e^c e^{t^2}.}$$

The application of $y(0) = -2$ gives $e^c = 2$, and hence $|y| = 2e^{t^2}$.

So $y = \pm 2e^{t^2}$, and the second application of the IC tells us that the negative version is correct.

However, my experience of marking exam scripts tells me that the modulus signs just disappear for no obvious reason! Then the IC gives $e^c = -2$ —— what?

My preferred safe route through a problem like this would be to use the following at the appropriate place.

$$y = e^{t^2+c} = Ae^{t^2}$$

where A is still arbitrary including its sign.

OK, this analysis has been far too wordy and disjointed, and therefore the best thing to do is for me to run this all again from scratch as a full analysis.

$$\frac{dy}{dt} = 2ty$$

$$\implies \int \frac{dy}{y} = \int 2t\,dt$$

$$\implies \ln|y| = t^2 + c$$

$$\implies |y| = e^{t^2+c} = e^c\, e^{t^2}$$

$$\implies y = Ae^{t^2}$$ replacement of the arbitrary constant

$$\implies y = -2e^{t^2}$$ using $A = -2$ from the initial conditions.

1.5 1st order linear equations

These equations fall into the general form,

$$\boxed{\frac{dy}{dt} + P(t)y = Q(t)}$$

where $P(t)$ and $Q(t)$ are given functions of t.

There is a general method for solving these equations, but let's consider the following example first as a taster.

Then we'll derive the general method, and consider various examples afterwards.

Example 1.4: Solve the equation

$$\boxed{\frac{dy}{dt} + \frac{2}{t}y = 5t^2.}$$

We'll multiply both sides by t^2 — rabbit-out-of-a-hat mathematics!

$$\boxed{t^2\frac{dy}{dt} + 2ty = 5t^4.}$$

Note: that the LHS is an exact derivative of t^2y. Hence,

$$\boxed{\frac{d}{dt}\left(t^2y\right) = 5t^4.}$$

We may now integrate both sides to obtain

$$\boxed{t^2y = t^5 + c} \qquad \Longrightarrow \qquad \boxed{y = t^3 + ct^{-2}}$$

where c is an arbitrary constant.

So why did that work? And how can we do it every time?

1.5.1 The Integrating Factor

The following derivation is for interest only.

We need to find an $F(t)$ to multiply the equation by to yield an exact derivative on the LHS. So,

$$\boxed{\frac{dy}{dt} + P(t)y = Q(t),} \quad \Longrightarrow \quad \boxed{F\frac{dy}{dt} + FPy = FQ} \quad \equiv \quad \boxed{\underbrace{F\frac{dy}{dt} + \frac{dF}{dt}y}_{(Fy)'} = FQ,}$$

Hence

$$\boxed{\frac{dF}{dt} = FP} \quad \Longrightarrow \quad \boxed{\frac{1}{F}\frac{dF}{dt} = P(t),}$$

which is of variables-separable type, and therefore

$$\boxed{\int \frac{dF}{F} = \int P(t)\,dt} \quad \Longrightarrow \quad \boxed{\ln|F| = c + \int P(t)\,dt} \quad \Longrightarrow \quad \boxed{F = Ae^{\int P(t)\,dt}.}$$

Note: we don't use A because it always cancels out and plays no role.

Thus, the **Integrating Factor** is

$$\boxed{F = e^{\int P(t)\,dt}.}$$

Example 1.4: revisited. Solve the equation

$$\boxed{\frac{dy}{dt} + \frac{2}{t}y = 5t^2.}$$

The Integrating Factor is

$$F = e^{\int (2/t)\,dt} = e^{2\ln t} = e^{\ln t^2} = t^2,$$

which is precisely the function by which we multiplied.

This method also yields a formula for the general solution. Using $F = e^{\int P(t)\,dt}$ we get,

$$\boxed{y' + Py = Q} \quad \Rightarrow \quad \boxed{Fy' + \underbrace{FP}_{F'}y = FQ} \quad \Rightarrow \quad \boxed{(Fy)' = FQ}$$

$$\Rightarrow \quad \boxed{Fy = c + \int FQ\,dt} \quad \Rightarrow \quad \boxed{y = \left[c + \int FQ\,dt\right]/F,}$$

but I would very definitely recommend remembering only the formula for F, and then proceeding as in the following examples.

Example 1.5: Solve the equation, $\boxed{y' + \frac{3y}{t} = \frac{2}{t^2}}$ subject to $y(1) = 2$.

The coefficient of y is $3/t$, so the Integrating Factor is

$$\boxed{e^{\int (3/t)\, dt} = e^{3 \ln |t|} = e^{\ln |t^3|} = t^3.}$$

Note: that this is one of the rare occasions when one doesn't need to worry about the modulus signs in a logarithm! So let us multiply the original ODE by t^3. We get

$$\boxed{t^3 y' + 3t^2 y = 2t.}$$

The left hand side is guaranteed to be an exact derivative of something. In this case it is the derivative of $t^3 y$. We get,

$$\boxed{(t^3 y)' = 2t} \quad \Longrightarrow \quad \boxed{t^3 y = t^2 + c} \quad \Longrightarrow \quad \boxed{y = \frac{1}{t} + \frac{c}{t^3}.}$$

The application of the initial condition, $y(1) = 2$, yields $c = 1$ and therefore the final solution is,

$$\boxed{y = \frac{1}{t} + \frac{1}{t^3}.}$$

Example 1.6: Solve the equation, $\boxed{y' + 2xy = 4xe^{-x^2},}$ subject to $y(0) = 1$.

OK, the right hand side looks horrible, but ignore that!

Given that the coefficient of y is $\boxed{2x}$ the Integrating Factor is, $\boxed{e^{\int 2x\,dx} = e^{x^2}.}$

On multiplying the ODE by the Integrating Factor we get,

$$\underbrace{e^{x^2}y' + 2xe^{x^2}y}_{(e^{x^2}y)'} = 4x.$$

So we have,

$$\boxed{\left(e^{x^2}y\right)' = 4x} \quad \Longrightarrow \quad \boxed{e^{x^2}y = 2x^2 + c} \quad \Longrightarrow \quad \boxed{y = \left(2x^2 + c\right)e^{-x^2}.}$$

Application of the Initial Condition, $y(0) = 1$, yields $c = 1$. Hence the required solution is,

$$\boxed{y = \left(2x^2 + 1\right)e^{-x^2}.}$$

Example 1.7: Solve the ODE, $\boxed{(\cot t)\, y' + y = \cot t \;\; \cos t}$ subject to $y(0) = 1$.

Note: Our theory of Integrating Factors relies on having a unit coefficient of y'. So we need to sort that out first. Therefore we'll multiply throughout by $\tan t$.

$$\boxed{y' + (\tan t)\, y = \cos t.}$$

The integrating factor is,

$$\boxed{F = e^{\int \tan t \, dt} = e^{-\int (-\sin t/\cos t)\, dt} = e^{-\ln \cos t} = 1/\cos t.}$$

Note how the integrand was manipulated to get it into an f'/f form, which necessitated the use of two minus signs.

Multiplication by the Integrating Factor yields, $\boxed{(\sec t)\, y' + (\tan t \sec t)\, y = 1.}$

Again the LHS is an exact derivative, by design, hence,

$$\boxed{\Big((\sec t)\, y\Big)' = 1} \quad \Longrightarrow \quad \boxed{(\sec t)\, y = t + c} \quad \Longrightarrow \quad \boxed{y = (t + c)\cos t.}$$

Application of the initial condition, $y(0) = 1$, gives $c = 1$ and hence the final solution is,

$$\boxed{y = (t + 1)\cos t.}$$

Example 1.8: Solve the equation, $\boxed{(\cot t)\, y' - y = \cot t\, \cos t}$ subject to $y(0) = 1$.

The same as Ex. 1.7 except for the replacement of a plus by a minus. Again multiply by $\tan t$:

$$\boxed{y' - (\tan t)\, y = \cos t.}$$

The integrating factor is, $\boxed{F = e^{\int -\tan t\, dt} = e^{\int (-\sin t/\cos t)\, dt} = e^{\ln \cos t} = \cos t.}$

Multiplication by F yields, $\boxed{(\cos t)\, y' - (\sin t)\, y \;=\; \cos^2 t \;=\; \tfrac{1}{2}(1 + \cos 2t),}$ using a multiple angle formula. Hence,

$$\Big[(\cos t)\; y\Big]' \;=\; \tfrac{1}{2}(1 + \cos 2t)$$

$$\Longrightarrow \quad (\cos t)\, y \;=\; \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t + c \;=\; \tfrac{1}{2}(t + \sin t \cos t) + c \qquad \text{multiple angles again}$$

$$\Longrightarrow \quad y \;=\; \tfrac{1}{2}(t \sec t + \sin t) + c \sec t$$

But $y(0) = 1 \Longrightarrow c = 1$ and hence the final solution is,

$$\boxed{y = \tfrac{1}{2}\Big[(t + 2)\sec t + \sin t\Big].}$$

So the change of one sign between Examples 1.7 and 1.8 results in two very different solutions.

And finally:

- It is important to be able to distinguish between the two types of equation in order to know which method of solution is needed. Is it $\boxed{y' = f(y)g(t)}$ or is it $\boxed{y' + P(t)y = Q(t)}$?
- The equation $\boxed{y' = g(t)\, y}$ belongs to both categories!
- Both the separation of variables and Integrating Factors techniques reduce down to doing some integration of the kind we did in Maths 1.
- The proof for the Integrating Factor will not be examined, but at least we now know where it comes from!
- In the examples I have concentrated mainly on some examples which have, in my experience, caused students some issues. So I have tried to find safe ways of approaching these slightly tricky cases.

- The rest of the ODEs section will now consist of methods for solving linear constant-coefficient ODEs, primarily because these are ubiquitous in Engineering and Science.