1.6 Solution of homogeneous, linear, constant-coefficient ODEs

The most general n^{th} order **homogeneous ODE** may be written in the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0,$$

where all the a_i coefficients are real and $a_n \neq 0$.

Given that $a_n \neq 0$, we could divide by a_n to yield an n^{th} order ODE with $d^n y/dt^n$ having a unit coefficient.

Its counterpart, the most general n^{th} order **inhomogeneous ODE**, has the form,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = F(t),$$

where $F(t)$ is a given function, and it is called the **Forcing term**. (Sometimes this is called a heterogeneous term.)

We shall be studying both sorts of ODE.

Systems of ODEs will be deferred to the end of the Matrix section.

Note: Such ODEs and systems of such ODEs are ubiquitous. They form the core of the modelling of physical systems. Examples include the following vibrating structures:

- Aircraft
- Bridges
- Buildings
- Musical instruments

Other applications in which ODEs may arise include,

- Fluid flows (especially boundary layers)
- Heat transfer
- Modelling of hydraulic/electrical circuits
- Planetary orbits

Note: I intend to develop a unified approach to solving these equations.

I will present very little theory, but will develop the topic and approach using a suitably-organised set of examples.

First, we'll begin with homogeneous ODEs, and then matters will be extended to inhomogeneous ODEs.

Example 1.1: Solve the ODE, $\boxed{y' + ay = 0.}$

This is both a separation-of-variables ODE and a first-order-linear ODE, but that won't help us solve higher order ODEs.

Obviously $y = 0$ is a solution, but that's boring. We need something better than that.

The ODE is

$$\boxed{y' = -ay,}$$

which means that both y and y' must involve the same function, even if their coefficients are different.

So y must be $\boxed{e^{\lambda t},}$ where λ is a constant which is chosen to satisfy the equation.

So let $y = Ae^{\lambda t}$ where A is arbitrary. We get,

$$A\lambda e^{\lambda t} = -aAe^{\lambda t} \qquad \Longrightarrow \qquad A(\lambda + a)e^{\lambda t} = 0.$$

We must have $(\lambda + a) = 0$, or $\lambda = -a$. Therefore the solution is,

$$\boxed{y = Ae^{-at}.}$$

The solution is

$$\boxed{y = Ae^{-at}.}$$

If the Initial Condition were $y(0) = 2$ then $A = 2$ and hence $\boxed{y = 2e^{-at}.}$

If the Initial Condition were $y(1) = 2$, then $A = 2e^{a}$ and hence $\boxed{y = 2e^{-a(t-1)}}$ — do check that out!

In general, $y(b) = c \quad \Longrightarrow \quad \boxed{y = ce^{-a(t-b)}.}$

Example 1.2: Solve the ODE, $\boxed{y'' + 3y' + 2y = 0.}$

Given our experience with Example 1.1 we shall use $y = e^{\lambda t}$. We obtain,

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$$

$$\implies \quad (\lambda^2 + 3\lambda + 2)e^{\lambda t} = 0$$

$$\implies \quad \underbrace{\lambda^2 + 3\lambda + 2 = 0}_{\text{Auxiliary Equation}} \qquad \text{since } e^{\lambda t} \text{ cannot be zero}$$

$$\implies \quad (\lambda + 1)(\lambda + 2) = 0 \qquad \text{by factorisation}$$

$$\implies \quad \lambda = -1, -2.$$

Two choices for λ — which one to favour, or use both? We have,

$$\boxed{y = Ae^{-t} + Be^{-2t},}$$

where A and B are arbitrary constants.
We may add these together because the ODE is linear — check by substitution.

Note that the equation labelled, **Auxiliary Equation**, is also known as the **Indicial Equation** or even as the **Characteristic Equation**.

In practice we would have two boundary/initial conditions to satisfy in order to find A and B.

$$\boxed{y = Ae^{-t} + Be^{-2t}}$$

An example of an Initial Value Problem might be:

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \Rightarrow \begin{cases} A + B = 0 \\ -A - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \end{cases} \Rightarrow \boxed{y = e^{-t} - e^{-2t}.}$$

An example of a Boundary Value Problem is

$$\begin{cases} y(0) = 1 \\ y(1) = 0 \end{cases} \Rightarrow \begin{cases} A + B = 1 \\ Ae^{-1} + Be^{-2} = 0 \end{cases} \Rightarrow \begin{cases} A = 1/(1-e) \\ B = -e/(1-e) \end{cases} \Rightarrow \boxed{y = \frac{e^{-t} - e^{1-2t}}{1-e}.}$$

The final solution needed a few lines of manipulation to get it into such a compact form.

Again, it's worth checking this.

Note that it is quite easy to write down the auxiliary equation because the coefficients of powers of λ correspond to the coefficients of the respective derivatives of y:

$$y'' + 3y' + 2y = 0 \quad \Longrightarrow \quad \lambda^2 + 3\lambda + 2 = 0.$$

This also works the other way:

$$3\lambda^4 - 2\lambda^2 + \lambda + 6 = 0 \quad \Longrightarrow \quad 3y'''' - 2y'' + y' + 6y = 0.$$

Therefore an homogeneous ODE has a unique auxiliary equation and vice versa; knowledge of one means that we have knowledge of the other.

Example 1.3: Solve the equation, $\boxed{y''' - 2y'' - y' + 2y = 0.}$

We may now go a litle faster. Use of $y = e^{\lambda t}$ gives,

$$\begin{aligned} \lambda^3 - 2\lambda^2 - \lambda + 2 &= (\lambda - 2)(\lambda^2 - 1) \\ &= (\lambda - 2)(\lambda + 1)(\lambda - 1) = 0, \end{aligned}$$

for which the roots are $\lambda = 2, 1, -1$.

The solution is

$$\boxed{y = Ae^{2t} + Be^{t} + Ce^{-t},}$$

where A, B and C are arbitrary, unless we have boundary/initial conditions.

Note: Now it gets really boring. Nothing new happens when the λ-values are real and different.

For example, in the case of a 5th order ODE there will be a fifth order Auxiliary equation to solve, five boundary conditions to satisfy and hence five algebraic equations to solve for those five unknown constants. Nasty.

So what's next?

Example 1.4: Solve the equation, $\boxed{y'' + 2y' = 0.}$

We have, $\lambda^2 + 2\lambda = 0 \quad \Longrightarrow \quad \lambda(\lambda + 2) = 0 \quad \Longrightarrow \quad \lambda = 0, -2.$

Do not worry about $\lambda = 0$, just treat it like all other λ-values:

The solution is,

$$y = Ae^{0t} + Be^{-2t} = \boxed{A + Be^{-2t}.}$$

So $\lambda = 0$ is equivalent to $y =$ constant.

Another example:

$$y''' + 3y'' + 2y' = 0$$

$$\Rightarrow \quad \lambda^3 + 3\lambda^2 + 2\lambda = 0$$

$$\Rightarrow \quad \lambda(\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \quad \lambda = 0, -1, -2$$

$$\Rightarrow \quad \boxed{y = A + Be^{-t} + Ce^{-2t}.}$$

Example 1.5: Solve the ODE, $\boxed{y'' + 9y = 0.}$

The auxiliary equation is $\boxed{\lambda^2 + 9 = 0,}$ hence

$$\lambda^2 = -9 \qquad \Longrightarrow \qquad \boxed{\lambda = \pm 3j.}$$

The auxiliary equation has purely imaginary roots. Is it a problem? No, not at all. Proceed normally...

$$\boxed{y = Ae^{3jt} + Be^{-3jt}.}$$

But don't we wish to have real solutions?

How about this:

$$\begin{aligned} y &= Ae^{3jt} + Be^{-3jt} \\ &= A(\cos 3t + j \sin 3t) + B(\cos 3t - j \sin 3t) \\ &= (A + B) \cos 3t + (Aj - Bj) \sin 3t \\ &= \boxed{C \cos 3t + D \sin 3t,} \end{aligned}$$

where $C = A + B$ and $D = (A - B)j$.

If we wish C and D to be real, then A and B must be complex conjugates of one another.

Note 1: There are two main ways of writing down the solution:

$$\boxed{y = Ae^{3jt} + Be^{-3jt}} \qquad \text{and} \qquad \boxed{y = C\cos 3t + D\sin 3t.}$$

The latter is *usually* chosen, but not always.

Note 2: They give the same solution when BCs/ICs are specified. Let $y(0) = 1$ and $y'(0) = 0$:

On taking the complex exponential form, $y = Ae^{3jt} + Be^{-3jt}$, we have

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \Rightarrow \begin{cases} A + B = 1 \\ 3j(A - B) = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \end{cases} \Rightarrow \boxed{y = \tfrac{1}{2}\left[e^{3jt} + e^{-3jt}\right] = \cos 3t.}$$

On taking the sinusoidal form, $y = C\cos 3t + D\sin 3t$, we have

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \Rightarrow \begin{cases} C = 1 \\ 3D = 0 \end{cases} \Rightarrow \boxed{y = \cos 3t.}$$

So both forms of solution will yield the correct real answer, but the one involving sinusoids is quicker.

Example 1.6: Solve the ODE, $\boxed{y'' + 4y' + 13y = 0.}$

Hence, $\lambda^2 + 4\lambda + 13 = 0 \implies (\lambda + 2)^2 + 9 = 0$ completing the square

$$\implies (\lambda + 2)^2 = -9$$

$$\implies \lambda + 2 = \pm 3j$$

$$\implies \lambda = -2 \pm 3j.$$

A complex conjugate pair of λ-values. Aaaargh! But again, just proceed normally:

$$\boxed{y = Ae^{(-2+3j)t} + Be^{(-2-3j)t}}$$

$$\implies \boxed{y = e^{-2t}\Big[Ae^{3jt} + Be^{-3jt}\Big]} \quad \text{or} \quad \boxed{y = e^{-2t}\Big[C\cos 3t + D\sin 3t\Big]}$$

In general, should $\lambda = a \pm bj$ then the general solution would be,

$$\boxed{y = e^{at}\Big[Ae^{bjt} + Be^{-bjt}\Big]} \quad \text{or} \quad \boxed{y = e^{at}\Big[C\cos bt + D\sin bt\Big].}$$

Solutions of this form are always associated with damped vibrating systems such as structures and stringed musical instruments. Hence a should be negative for these applications.

Example 1.7: Solve the equation, $\boxed{y'' + 4y' + 4y = 0.}$

This is the first really special case, a repeated root. We have,

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\implies \quad (\lambda + 2)^2 = 0$$

$$\implies \quad \lambda = -2, \, -2.$$

We have a repeated value for λ. But how do we treat such cases?

We can't use $y = Ae^{-2t}$ because a 2nd ODE needs two BCs/ICs.

But clearly e^{-2t} must play a role.

Let us experiment with this by factoring out the e^{-2t}. Hopefully this will show us how to deal with a repeated root.

$$\boxed{y'' + 4y' + 4y = 0.}$$

Let	$y = e^{-2t}z(t)$	
$\Longrightarrow$	$y' = e^{-2t}[z' - 2z]$	(product rule and tidying up)
$\Longrightarrow$	$y'' = e^{-2t}[z'' - 4z' + 4z]$	(product rule and more tidying up).

Substitution into the ODE gives,

$$\underbrace{e^{-2t}[z'' - 4z' + 4z]}_{y''} + \underbrace{4e^{-2t}[z' - 2z]}_{4y'} + \underbrace{4e^{-2t}[z]}_{4y} = 0.$$

All but one of these terms then cancel.

$$e^{-2t}[z'' - 4z' + 4z] + e^{-2t}[4z' - 8z] + e^{-2t}[4z] = 0.$$

We are left with,

$$e^{-2t}z'' = 0 \qquad \Longrightarrow \qquad z'' = 0.$$

Hence $z = A + Bt$, and therefore,

$$\boxed{y = (A + Bt)e^{-2t}.}$$

So we now have two arbitrary constants.

$$\boxed{y = (A + Bt)e^{-2t}.}$$

Note 1: Practical consequence.

- The constant term, A, corresponds to the first of the two repeated λ-values.
- The Bt term corresponds to the second λ-value.

Note 2: If we had $\lambda = -5, -5$ then $\boxed{y = (A + Bt)e^{-5t}}$ is the solution of $y'' + 10y' + 25y = 0$.

Likewise, $\boxed{y'' + 2cy' + c^2y = 0}$ $\Rightarrow$ $\boxed{\lambda = -c, -c}$ $\Rightarrow$ $\boxed{y = (A + Bt)e^{-ct}.}$

Note 3: The substitution, $y = e^{-2t}z(t)$, has been used here solely to determine what the solution is for a repeated value of λ.

Example 1.8: Solve the ODE, $\boxed{y''' + 3y'' + 3y' + y = 0.}$ Ah, binomial coefficients!

The auxiliary equation for this ODE is,

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \qquad \Longrightarrow \qquad (\lambda + 1)^3 = 0 \qquad \Longrightarrow \qquad \lambda = -1, -1, -1.$$

The auxiliary equation has a triple root and we have three instances of $\lambda = -1$. The solution is,

$$\boxed{y = (A + Bt + Ct^2)e^{-t}.}$$

- A corresponds to the 1st of the λ-values.
- Bt corresponds to the 2nd λ-value.
- Ct^2 corresponds to the 3rd λ-value.

$$\boxed{y''' + 3y'' + 3y' + y = 0.}$$

Note 1: If we had used $y = e^{-t}z(t)$ as a substitution, then ...

$$\begin{aligned}
& y = e^{-t}z(t) \\
\Longrightarrow \quad & y' = e^{-t}[z' - z] && \text{(product rule and tidying up)} \\
\Longrightarrow \quad & y'' = e^{-t}[z'' - 2z' + z] && \text{(product rule and more tidying up).} \\
\Longrightarrow \quad & y''' = e^{-t}[z''' - 3z'' + 3z' - z] && \text{(product rule and yet more tidying up).}
\end{aligned}$$

Substitution into the ODE gives,

$$\underbrace{e^{-t}\left[z''' - 3z'' + 3z' - z\right]}_{y'''} + \underbrace{3e^{-t}[z'' - 2z' + z]}_{3y''} + \underbrace{3e^{-t}[z' - z]}_{3y'} + \underbrace{e^{-t}[z]}_{y} = 0.$$

All that emerges from a monstrous number of cancellations is $z''' = 0$. Hence,

$$z = (A + Bt + Ct^2) \quad \Longrightarrow \quad \boxed{y = (A + Bt + Ct^2)e^{-t}.}$$

Note 2: All of this extends to more repetitions such as $(\lambda - 4)^6 = 0$, which will give the following solution,

$$\boxed{y = (\underbrace{A}_{1\text{st}} + \underbrace{Bt}_{2\text{nd}} + \underbrace{Ct^2}_{3\text{rd}} + \underbrace{Dt^3}_{4\text{th}} + \underbrace{Et^4}_{5\text{th}} + \underbrace{Ft^5}_{6\text{th}})e^{4t}.}$$

Example 1.9: Solve the ODE, $y'''' + 4y''' + 4y'' = 0$.

The auxiliary equation in this case is,

$$\lambda^4 + 4\lambda^3 + 4\lambda^2 = 0 \qquad \Longrightarrow \qquad \lambda^2(\lambda+2)^2 = 0 \qquad \Longrightarrow \qquad \lambda = 0, 0, -2, -2.$$

We have two pairs of repeated roots. The solution is,

$$y = \underbrace{(A+Bt)e^{0t}}_{\lambda = 0, 0} + \underbrace{(C+Dt)e^{-2t}}_{\lambda = -2, -2},$$

The two pairs of repeated roots are treated independently but in the usual way for repeated roots.

$$\boxed{y = (A+Bt) + (C+Dt)e^{-2t}.}$$

Example 1.10: If the auxiliary equation is $\boxed{(\lambda+3)^4(\lambda+1)(\lambda+1+2j)^2(\lambda+1-2j)^2\lambda^3=0,}$ then what is the solution of the equivalent ODE?

The roots of the auxiliary equation are,

$$\lambda=-3,-3,-3,-3, \qquad -1, \qquad -1\pm 2j,-1\pm 2j, \qquad 0,0,0,$$

and therefore the solution of the equivalent 12^{th} ODE is,

$$y = \underbrace{(A+Bt+Ct^2+Dt^3)e^{-3t}}_{\lambda=-3,-3,-3,-3} + \underbrace{Ee^{-t}}_{\lambda=-1} + \underbrace{\Big[(F+Gt)\cos 2t+(H+It)\sin 2t\Big]e^{-t}}_{\lambda=-1\pm 2j,-1\pm 2j}$$

$$+\underbrace{(J+Kt+Lt^2)}_{\lambda=0,0,0}.$$

Note: We have used,

$$\boxed{(\mathcal{F}+\mathcal{G}t)e^{(-1+2j)t}+(\mathcal{H}+\mathcal{I}t)e^{(-1-2j)t}} = \boxed{\Big[(F+Gt)\cos 2t+(H+It)\sin 2t\Big]e^{-t}.}$$

Should you be interested the ODE is

$$\boxed{\begin{aligned} y^{(12)}+17y^{(11)}+132y^{(10)}+628y^{(9)}+2026y^{(8)}+4750y^{(7)}+7860y^{(6)} \\ +8964y^{(5)}+6345y''''+2025y'''=0. \end{aligned}}$$

1.6.1 A checklist of examples.

Some examples of how the solution of an ODE is related to the roots of the auxiliary equation and their multiplicity:

Roots	Solution
$2,\ -2$	$Ae^{2t} + Be^{-2t}$
$2,\ 3$	$Ae^{2t} + Be^{3t}$
$2,\ 3,\ 4,\ 5,\ 10$	$Ae^{2t} + Be^{3t} + Ce^{4t} + De^{5t} + Ee^{10t}$
$2,\ 2$	$(A + Bt)e^{2t}$
$2,\ 2,\ 4$	$(A + Bt)e^{2t} + Ce^{4t}$
$2,\ 2,\ 4,\ 4$	$(A + Bt)e^{2t} + (C + Dt)e^{4t}$
$2,\ 2,\ 2,\ 2,\ 4,\ 4, 5$	$(A + Bt + Ct^2 + Dt^3)e^{2t} + (E + Ft)e^{4t} + Ge^{5t}$
$0,\ 2$	$A + Be^{2t}$
$0,\ 0,\ 0,\ 2$	$(A + Bt + Ct^2) + De^{2t}$

Roots	Solution
$\pm 2j$	$A\cos 2t + B\sin 2t$
$\pm 2j,\ \pm 5j, 2$	$A\cos 2t + B\sin 2t + C\cos 5t + D\sin 5t + Ee^{2t}$
$\pm 2j,\ \pm 2j$	$(A + Bt)\cos 2t + (C + Dt)\sin 2t$
$2 \pm 3j$	$e^{2t}(A\cos 3t + B\sin 3t)$
$2 \pm 3j,\ 2 \pm 4j$	$e^{2t}\,(A\cos 3t + B\sin 3t + C\cos 4t + D\sin 4t)$
$2 \pm 3j,\ 4 \pm 5j$	$e^{2t}\,(A\cos 3t + B\sin 3t) + e^{4t}\,(C\cos 5t + D\sin 5t)$
$2 \pm 3j,\ 2 \pm 3j$	$e^{2t}\,[(A + Bt)\cos 3t + (C + Dt)\sin 3t]$
$2 \pm 3j,\ 2 \pm 3j,\ 2 \pm 3j$	$e^{2t}\,[(A + Bt + Ct^2)\cos 3t + (D + Et + Ft^2)\sin 3t)]$

Note 1: There is no difference between how we treat zero, nonzero real and complex values of $\boldsymbol{\lambda}$.

Zero values are equivalent to a constant, then to $\boldsymbol{t}$, then $\boldsymbol{t^2}$ and so on depending on the multiplicity of $\boldsymbol{\lambda = 0}$.

Complex values may be used directly as a complex exponential, or else in terms of a real exponential multiplied by sinusoids.

Note 2: In all cases multiplicities in the values of $\boldsymbol{\lambda}$ are treated identically using increasing powers of $\boldsymbol{t}$. This happens independently of the presence of different values of $\boldsymbol{\lambda}$.