Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem Sheet 1 — Complex Numbers

Q1. Simplify the following:

(a) j^3, **(b)** j^4, **(c)** j^5, **(d)** j^{10}, **(e)** j^{2023}, **(f)** $(1+j)(2+j)$, **(g)** $(1+j)(1-j)$,
(h) $(2+j)(1+3j)+(2-j)(1-3j)$, **(i)** $(2+j)(2+3j)-(2-j)(2-3j)$, **(j)** $2/(1-j)$,
(k) $(3+j)/(4+3j)$, **(l)** $(1+j)^2$, **(m)** $(1+j)^{100}$.

A1. **(a)** $j^3 = -j$.

(b) $j^4 = 1$.

(c) $j^5 = j$.

(d) $j^{10} = j^6 = j^2 = -1$, on removing factors of j^4 since they are equal to 1.

(e) $j^{2023} = j^{4\times 505+3} = (j^4)^{505} \times j^3 = 1^{505} \times (-j) = -j$.

(f) $(1+j)(2+j) = 2 + j^2 + 2j + j = 1 + 3j$.

(g) $(1+j)(1-j) = 1 - j^2 + j - j = 2$.

(h) $(2+j)(1+3j)+(2-j)(1-3j) = [2+3j^2+j+6j] + [2+3j^2-j-6j] = -2$.

The answer is real because we are adding complex conjugates.

(i) $(2+j)(2+3j)-(2-j)(2-3j) = [4+3j^2+2j+6j] - [4+3j^2-2j-6j] = 16j$.
The answer is purely imaginary because we are subtracting complex conjugates.

(j) $\frac{2}{1-j} = \frac{2(1+j)}{(1-j)(1+j)} = \frac{2(1+j)}{2} = 1+j$.

(k) $\frac{3+j}{4+3j} = \frac{(3+j)(4-3j)}{(4+3j)(4-3j)} = \frac{1}{5}(3-j)$.

(l) $(1+j)^2 = 2j$.

(m) $(1+j)^{100} = (2j)^{50} = 2^{50}j^{50} = 2^{50}(-1)^{25} = -2^{50} = -1125899906842624$.
I doubt if your calculator would be able to compute this number!

Q2. Find $(1+\sqrt{3}j)^3$, and hence find $(1+\sqrt{3}j)^{60}$.

A2.

$$\begin{aligned}(1+\sqrt{3}j)^3 &= 1^3 \quad + \quad 3\times(\sqrt{3}j) \quad + \quad 3\times(\sqrt{3}j)^2 \quad + \quad (\sqrt{3}j)^3 \\ &= 1 + 3\sqrt{3}j - 9 - 3\sqrt{3}j \\ &= -8.\end{aligned}$$

Hence $(1+\sqrt{3}j)^{60} = (-8)^{20} = 2^{60}$.

Q3. Find the modulus and argument of each of the following complex numbers, and hence write them in complex exponential form.

(a) $2+3j$, **(b)** $-j$, **(c)** $40+9j$, **(d)** -5, **(e)** $1-100j$, **(f)** $1+\sqrt{3}j$.

A3. **(a)** $2+3j$.

We have $r = \sqrt{2^2+3^2} = \sqrt{13}$. Hence $z = \sqrt{13}e^{j\theta}$ where $\tan\theta = \frac{3}{2}$ and $0 < \theta < \pi/2$, i.e. $\theta = 0.982794$.

(b) $-j$.
We have $r = 1$ and $\theta = 3\pi/2$ or $-\pi/2$. Hence $-j = e^{3\pi j/2}$.

(c) $40 + 9j$.
We have $r = \sqrt{40^2+9^2} = 41$. Hence $z = 41e^{j\theta}$ where $\tan\theta = \frac{9}{40}$ and $0 < \theta < \pi/2$, i.e. $\theta = 0.221314$.

(d) -5.
Here, $z = 5e^{j\pi}$ or $z = 5e^{-j\pi}$.

(e) $1 - 100j$.
We have $z = \sqrt{10001}e^{j\theta}$ where $\tan\theta = -100$ and $-\frac{1}{2}\pi < \theta < 0$, i.e. $\theta = -1.560797$.

(f) $1 + \sqrt{3}j$.
We have $r = 2$ and $\tan\theta = \sqrt{3}$, which implies that $\theta = \pi/3$. Hence $(1+\sqrt{3}j) = 2e^{\pi j/3}$.

Q4. Convert the following numbers from complex exponential form to Cartesian form and write in as simple form as possible.

(a) $e^{j\pi/3}$, **(b)** $4e^{2\pi j/3}$, **(c)** $\sqrt{2}e^{3\pi j/4}$.

A4. **(a)** $e^{j\pi/3}$.
This is $\cos\frac{1}{3}\pi + j\sin\frac{1}{3}\pi = \frac{1}{2} + \frac{\sqrt{3}}{2}j$.

(b) $4e^{2\pi j/3}$.
This is $4(\cos\frac{2}{3}\pi + j\sin\frac{2}{3}\pi) = 4(-\frac{1}{2} + \frac{\sqrt{3}}{2}j) = -2 + 2\sqrt{3}j$.

(c) $\sqrt{2}e^{3\pi j/4} = -1 + j$.

Q5. Use de Moivre's theorem to find $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. [Note that $\cos 4\theta$ may be written in terms of cosines only or sines only; find both forms.] Also find $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Use your result for $\sin 5\theta$ to find an analytical expression for $\sin 36°$.

A5. On using

$$e^{4\theta j} = \left(e^{\theta j}\right)^4$$

in Euler's form,

$$\cos 4\theta + j\sin 4\theta = \left(\cos\theta + j\sin\theta\right)^4,$$

the solutions may be found to be,

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \qquad \sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta),$$

or

$$\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1 \qquad \sin 4\theta = 4\sin\theta\cos\theta(2\cos^2\theta - 1).$$

For the second part of the question we have,

$$\cos 5\theta + j\sin 5\theta = \left(\cos\theta + j\sin\theta\right)^5,$$

The expansion of the right hand side is,

$$c^5 + 5c^4(sj) + 10c^3(s)^2 + 10c^2(sj)^3 + 5c(sj)^4 + (s)^5,$$

where I have used c and s as shorthand for $\cos\theta$ and $\sin\theta$, respectively. After separating out the real and imaginary parts we get,

$$(c^5 - 10c^3s^2 + 5cs^4) + j(5c^4s - 10c^2s^3 + s^5).$$

Hence

$$\cos 5\theta = \cos\theta\Big(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta\Big)$$

and

$$\sin 5\theta = \sin\theta\Big(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta\Big).$$

Both sets of terms in the brackets could be tidied further to give expressions solely in terms of cosines or solely in terms of sines.

We may rewrite the above expression for $\sin 5\theta$ as,

$$\sin 5\theta = \Big(16\sin^4\theta - 20\sin^2\theta + 5\Big)\sin\theta.$$

If we now set $\theta = 36^\circ$ then $\sin 5\theta = \sin 180^\circ = 0$. Therefore

$$16\sin^4\theta - 20\sin^2\theta + 5 = 0.$$

This is a quadratic equation for $\sin^2\theta$ and hence,

$$\sin^2\theta = \frac{5 \pm \sqrt{5}}{8}.$$

Hence,

$$\sin\theta = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}},$$

This means that,

$$\sin\theta = \pm 0.587785,\ \pm 0.951056.$$

Of these four choices we have $\sin 36^\circ = 0.587785$.

Was that useful? Well, only if you have a 1970s calculator with the square root as its most advanced function! Yes, I did have one of those....

Q6. Evaluate the following roots of complex numbers:

(a) $(5+12j)^{1/2}$, **(b)** $(-16)^{1/4}$, **(c)** $1^{1/5}$, **(d)** $(-1)^{1/100}$,
(e) $(1+2j)^{2/7}$, **(f)** $(336+527j)^{1/4}$, **(g)** $(2j)^{1/2}$, **(h)** $(-15+8j)^{3/5}$.

A6. **(a)** Here $5 + 12j = 13e^{j\theta}$ and $13e^{j(\theta+2\pi)}$ where $\tan\theta = \frac{12}{5}$ and $0 < \theta < \frac{1}{2}\pi$, i.e. $\theta = 1.176005$.

Hence $(5+12j)^{1/2} = \sqrt{13}e^{j\theta/2}$ or $\sqrt{13}e^{j(\pi+\theta/2)} = \sqrt{13}e^{0.588003\,j}$ or $\sqrt{13}e^{3.729595\,j}$.

(b) We write: $-16 = 16e^{j\pi}$, $16e^{j3\pi}$, $16e^{j5\pi}$ and $16e^{j7\pi}$.

Note that it is necessary to write -1 in these four different forms because we will be finding the fourth roots. Now we can take the fourth roots to obtain:

$$(-16)^{1/4} = 2e^{j\pi/4},\quad 2e^{j3\pi/4},\quad 2e^{j5\pi/4}\quad \text{and}\quad 2e^{j7\pi/4}.$$

We could write this as,

$$(-16)^{1/4} = \pm\sqrt{2} \pm \sqrt{2}j,$$

where all four sign possibilities are allowed, and given that $e^{j\pi/4} = (1+j)/\sqrt{2}$.

(c) The solutions are: e^{0j}, $e^{j2\pi/5}$, $e^{j4\pi/5}$, $e^{j6\pi/5}$ and $e^{j8\pi/5}$. Note that $e^{0j} = 1$.

(d) Since $-1 = e^{j(2n+1)\pi}$ for $n = 0, 1, \cdots, 99$, the solutions take the form $e^{j(2n+1)\pi/100}$ for $n = 0, 1, \ldots, 99$.

(e) First multiply out the square: $(1+2j)^{2/7} = [(1+2j)^2]^{1/7} = [-3+4j]^{1/7} = [5e^{j(\theta+2n\pi)}]^{1/7}$ where $n = 0, 1, 2, \cdots, 6$, $\tan\theta = -\frac{4}{3}$ and $\frac{1}{2}\pi < \theta < \pi$. Actually $\theta = 2.2143$ radians.

Hence $(1+2j)^{2/7} = 5^{1/7}e^{j(\theta+2n\pi)/7}$ for $n = 0, 1, \cdots, 6$.

(f) Here, if $z = 336 + 527j$, then $|z| = \sqrt{336^2 + 527^2} = 625$, and $\arg(z) = 1.0032$ radians. Note that the fourth root of 625 is 5.

Hence

$$z^{1/4} = 5e^{(1.0032+2\pi n)j/4}, \quad \text{for} \quad n = 0, 1, 2, 3.$$

(g) OK, this one appears easier than many of the preceding ones, but this one arises a little more often than most.

We write $2j = 2e^{((\pi/2)+2n\pi)j}$, where $n = 0, 1$. That is, we have

$$2j = 2e^{j\pi/2},\ 2e^{j5\pi/2}.$$

Hence,

$$(2j)^{1/2} = \sqrt{2}e^{j\pi/4},\ \sqrt{2}e^{j5\pi/4}.$$

When we substitute in $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we get,

$$(2j)^{1/2} = 1+j,\ -1-j \quad = \quad \pm(1+j).$$

(h) The modulus of $-15+8j$ is 17, yes, $(8, 15, 17)$ is a Pythagorean triple. The argument is $\tan^{-1}(-8/15)$ which is $\theta = 2.651635$, a second quadrant value. So we write,

$$-15 + 8j = 17e^{(\theta+2n\pi)j} \quad \text{for} \quad n = 0, 1, 2, 3, 4.$$

Hence,

$$(-15+8j)^{3/5} = 17^{3/5}e^{3(\theta+2n\pi)j/5} \quad \text{for} \quad n = 0, 1, 2, 3, 4.$$

I haven't sketched any of these final solutions, but it is generally a good idea to do so in order to see where they lie in the complex plane.

Q7. Solve the following quadratic equations and plot the roots in the Argand diagram (or the complex plane). Find the modulus and argument of each root, and hence write them in complex exponential form.

(a) $x^2 + 2 = 0$ **(b)** $y^2 + 2jy - 2 = 0$ **(c)** $z^2 + 2\sqrt{2}z + 2 - 4j = 0$.

A7. (a) This one should be fairly straightforward: $x = \pm\sqrt{2}j$. Hence $x = \sqrt{2}e^{\pm\pi j/2}$.

(b) Using the traditional formula for solving quadratic equations we obtain

$$y = \frac{-2j \pm \sqrt{-4+8}}{2} = -j \pm 1.$$

For the complex exponential form: $|y| = \sqrt{2}$ and $\arg(y) = \frac{5}{4}\pi$ or $\frac{7}{4}\pi$, and therefore the answers are

$$y = \sqrt{2}e^{\frac{5}{4}\pi j} \quad \text{and} \quad y = \sqrt{2}e^{\frac{7}{4}\pi j}.$$

Note that these may also be written in the form

$$y = \sqrt{2}e^{-\frac{3}{4}\pi j} \quad \text{and} \quad y = \sqrt{2}e^{-\frac{1}{4}\pi j}.$$

(c) Again we obtain

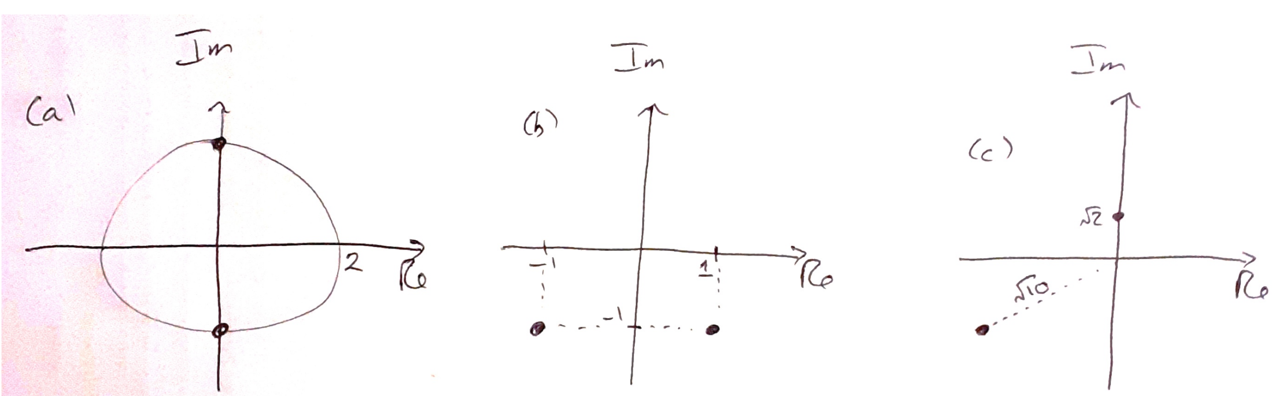
$$\begin{aligned} z &= \tfrac{-2\sqrt{2}\pm\sqrt{8-8+16j}}{2} = -\sqrt{2} \pm \sqrt{(4j)} = -\sqrt{2} \pm 2\sqrt{j} \\ &= -\sqrt{2} \pm 2\Big(\tfrac{1}{\sqrt{2}} + j\tfrac{1}{\sqrt{2}}\Big) \qquad \text{using the correct values for } \sqrt{j} \\ &= \sqrt{2}[-1 \pm (1+j)] = \quad \sqrt{2}(-2-j) \qquad \text{or} \qquad \sqrt{2}\,j. \end{aligned}$$

For solutions written in complex exponential form....

For the first root, $z = \sqrt{2}(-2-j)$, $|z| = \sqrt{10}$ and $\arg(z) = \theta$ where $\tan\theta = \frac{1}{2}$ and $\pi < \theta < \frac{3}{2}\pi$, i.e. $\theta = 3.605240$. Hence $z = \sqrt{10}e^{3.605240\,j}$. Note that we could also have had $\theta = -2.677945$, which is in the same direction in the Argand diagram.

For the second root, $z = \sqrt{2}j$, $|z| = \sqrt{2}$ and $\arg(z) = \frac{1}{2}\pi$. Hence $z = \sqrt{2}e^{j\pi/2}$.

Sketches for the solutions for these three question parts are given below.



Q8. When sketched in the complex plane, the three complex numbers, $z_1 = 1+j$, $z_2 = 3+j$ and $z_3 = 2$, clearly form a right-angled triangle. Multiply each of these numbers by j and sketch the results. How has the original triangle been transformed? Now multiply z_1, z_2 and z_3 by $1-j$; what is the nature of the transformation this time?

A8. We are given that,

$$z_1 = 1+j, \qquad z_2 = 3+j \qquad \text{and} \qquad z_3 = 2.$$

These are given as the black points in the Figure below, and the triangle is shaded very lightly simply to emphasize its shape.

Multiplication by j yields,

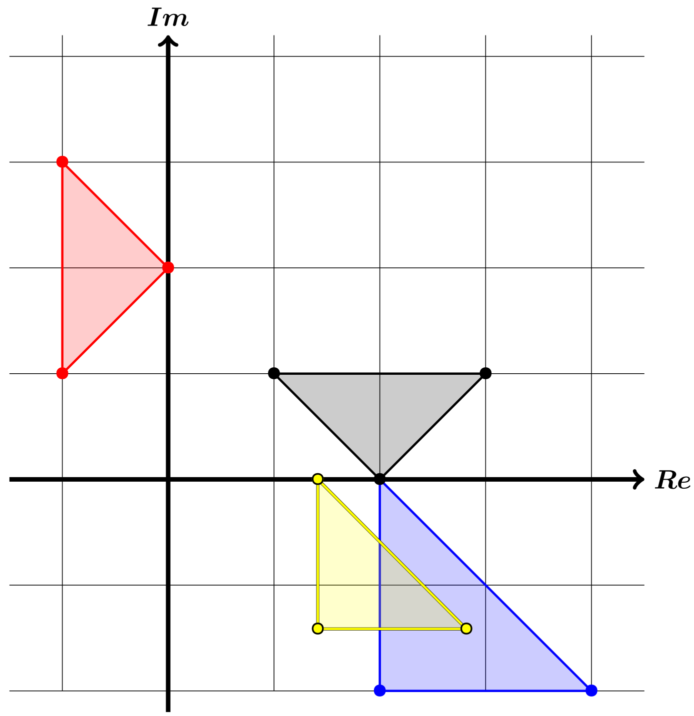
$$jz_1 = -1 + j, \qquad jz_2 = -1 + 3j \qquad \text{and} \qquad jz_3 = 2j.$$

These are included in the flgure as red points.

Finally, if the original points are multiplied by $(1 - j)$ then we obtain,

$$(1 - j)z_1 = 2, \qquad (1 - j)z_2 = 4 - 2j \qquad \text{and} \qquad (1 - j)z_3 = 2 - 2j.$$

These are depicted in blue.



So multiplication by j (i.e. by $e^{j\pi/2}$) is equivalent to rotation about the origin by the angle, $\pi/2$ (i.e. the black triangle to the red triangle). This is an anticlockwise rotation without a change in scale.

And mutiplication by $(1 - j)$ (i.e. by $\sqrt{2}e^{-j\pi/4}$) is equivalent to a rotation of $-\pi/4$ and a multiplication by $\sqrt{2}$. This is a clockwise rotation and a scaling factor of $\sqrt{2}$.

The yellow triangle is the result of mutiplying all of the points in the grey triangle by $e^{-j\pi/4}$, which may be seen as a rotation of $-\pi/4$ without a change of scale. Then the multiplication of these values by $\sqrt{2}$ yields the blue triangle.

The following questions are for interest only and are definitely not examinable. If you are bored, then do have a go at them!

Q9. de Moivre's theorem is used to express $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$ when n is an integer. The equivalent for the hyperbolic sine and cosine is not quite as easy to write down. However, write $\cosh\theta$ in terms of e^{θ} and $e^{-\theta}$ and square both sides of the equation. Hence find $\cosh 2\theta$ in terms of $\cosh\theta$.

Further, can you think of a quick way of determining a simple expression for $\sinh 2\theta$ in terms of $\sinh\theta$ and $\cosh\theta$? Also, find an expression for $\sinh 3\theta$ in terms of powers of $\sinh\theta$, and an expression for $\cosh 3\theta$ in terms of powers of $\cosh\theta$.

A9. We start with the definition of cosh in terms of exponentials,

$$\cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}).$$

Square this to get,

$$\begin{aligned} \cosh^2\theta &= \tfrac{1}{4}(e^{\theta} + e^{-\theta})^2 \\ &= \tfrac{1}{4}(e^{2\theta} + 2 + e^{-2\theta}) \\ &= \tfrac{1}{2}\cosh 2\theta + \tfrac{1}{2} \\ \Longrightarrow \qquad \cosh 2\theta &= 2\cosh^2\theta - 1. \end{aligned}$$

We start again with,

$$\begin{aligned} \sinh 2\theta &= \tfrac{1}{2}(e^{2\theta} - e^{-2\theta}) \\ &= \tfrac{1}{2}(e^{\theta} - e^{-\theta})(e^{\theta} + e^{-\theta}) \\ &= 2\sinh\theta\,\cosh\theta. \end{aligned}$$

Here I have noticed that $e^{2\theta} - e^{-2\theta}$ is the difference of two squares, and therefore I have reduced it the usual way (i.e. using $a^2 - b^2 = (a-b)(a+b)$).

An alternative might have been to say that, given the close similarity between trigonometric results and hyperbolic results, we should start with $\sinh\theta\,\cosh\theta$ (since $\sin 2\theta = 2\sin\theta\,\cos\theta$), substitute in the definitions of the two functions in terms of exponentials, and then see what happens. Either of these ways is fine.

Best to begin with,

$$\sinh^3\theta = \Big[\tfrac{1}{2}(e^{\theta} - e^{-\theta})\Big]^3.$$

This becomes,

$$\sinh^3\theta = \tfrac{1}{8}\Big[e^{3\theta} - 3e^{\theta} + 3e^{-\theta} - e^{-3\theta}\Big].$$

And so,

$$\sinh^3\theta = \tfrac{1}{4}\sinh 3\theta - \tfrac{3}{4}\sinh\theta.$$

After rearrangement the final answer is,

$$\sinh 3\theta = 4\sinh^3\theta + 3\sinh\theta.$$

In a very similar fashion we get,

$$\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta.$$

Q10. Leibniz, he of the invention-of-calculus fame, came late to complex numbers. He found that

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6},$$

but no-one was surprised! How does one prove this without finding the square roots of complex numbers?

A10. So we may possibly guess at the outset that the two terms on the left hand side are complex conjugates of one another. Specifically, $(1+\sqrt{3}j)$ and $(1-\sqrt{3}j)$ are conjugates. We can imagine where they are in the complex plane, and when we take their square roots (assuming that it's the "positive" ones, then the two main square roots will be conjugates of one another. Hence the final answer will be real. That's a good start.

Shall we use the earlier tricks to find the square roots, or is there a better way? Well, we could bludgeon our way through, but I spotted the 1 and the $\sqrt{3}$ coefficients. These aren't too far away from $\frac{1}{2}$ and $\frac{1}{2}\sqrt{3}$, which are the cosine and sine of 60° or, better still, $\pi/3$.

So,

$$1+\sqrt{3}\,j=2e^{j\pi/3}\quad\text{and}\quad 1-\sqrt{3}\,j=2e^{-j\pi/3}.$$

On taking the respective square roots (where we halve the arguments) we get,

$$(1+\sqrt{3}\,j)^{1/2}=\sqrt{2}\,e^{j\pi/6}\quad\text{and}\quad(1-\sqrt{3}\,j)^{1/2}=\sqrt{2}\,e^{-j\pi/6}$$

or

$$(1+\sqrt{3}\,j)^{1/2}=\sqrt{2}\,(\cos\tfrac{1}{6}\pi+j\sin\tfrac{1}{6}\pi)\quad\text{and}\quad(1-\sqrt{3}\,j)^{1/2}=\sqrt{2}\,(\cos\tfrac{1}{6}\pi-j\sin\tfrac{1}{6}\pi).$$

Adding these together means that the imaginary terms cancel, leaving us with,

$$(1+\sqrt{3}\,j)^{1/2}+(1-\sqrt{3}\,j)^{1/2}=2\sqrt{2}\cos\tfrac{1}{6}\pi=2\sqrt{2}(\sqrt{3}/2)=\sqrt{6}.$$

Yet there is another way of doing this without resorting to complex exponentials, and all that we have to do is to square the left hand side to find that it is precisely equal to 6. Here we go: let

$$z=\sqrt{1+\sqrt{3}j}+\sqrt{1-\sqrt{3}j}.$$

Hence,

$$\begin{aligned}z^2&=(1+\sqrt{3}j)+2\sqrt{(1+\sqrt{3}j)(1-\sqrt{3}j)}+(1-\sqrt{3}j)\\&=2+2\sqrt{(1+3)}\\&=6.\end{aligned}$$

Hence $z=\sqrt{6}$. Yes, one ought to be a tiny bit more careful about some of these plusses and minuses....

Q11. Express $z=\frac{1}{2}(\sqrt{3}+1)+\frac{1}{2}(\sqrt{3}-1)j$ in complex exponential form. Now find the argument in terms of degrees, and hence find the first integer power of z for which it is a real value.

Answer: Somehow this feels related to the previous questions, but it isn't.

The modulus of z is given by

$$|z|^2=\Big(\frac{\sqrt{3}+1}{2}\Big)^2+\Big(\frac{\sqrt{3}-1}{2}\Big)^2=(2+\sqrt{3})+(2-\sqrt{3})=4.$$

Hence $|z| = 2$.

We'll need to use the calculator to find arg(z):

$$\theta = \arg(z) = \tan^{-1}\Big(\frac{\sqrt{3}-1}{\sqrt{3}+1}\Big) = \tan^{-1} 0.267949 = 0.261799.$$

If we convert θ from radians to degrees we get $\theta = 15^\circ$, precisely. But this is $1/24$th of a full 360° circle, and hence

$$z = 2e^{2\pi j/24}.$$

So if we raise this to the 24th power, then

$$z^{24} = 2^{24}e^{2\pi j} = 2^{24} = 16\,777\,216.$$

All of this means that the original real and imaginary parts of z are the cosine and sine of 15°, and that is what motivated this question.

Q12. Some very weird ones. Not examinable. You may need to resort to some lateral thinking...

(a) $\ln(-1)$, **(b)** $\ln j$, **(c)** j^j, **(d)** $y = \cos^{-1} 2$.

A12. (a) $\ln(-1)$,

If $x + yj = \ln(-1)$, then $e^{x+yj} = -1$. But we already know that $-1 = e^{(\pi+2n\pi)j}$ for integer values of n. This leads to

$$e^{x+yj} = e^{(\pi+2n\pi)j} \qquad \Rightarrow x + yj = (\pi + 2n\pi)j,$$

and therefore $x = 0$ and and $y = \pi + 2n\pi$. So we can say that,

$$\ln(-1) = (\pi + 2n\pi)j \qquad n = 0, \pm1, \pm2, \cdots,$$

and we appear to have an infinite number of imaginary numbers which all claim to be $\ln(-1)$. That must make some kind of sense, as the answer couldn't have been real!

Additional bit: There is the complex numbers Christmas joke:

What is the yule log of -1? Answer: An imaginary mince pi.

This may be expressed in mathematical form by setting $n = -1$ in the above solution to give,

$$\ln(-1) = j(-\pi).$$

(b) $\ln j$,

In like manner, set $x + yj = \ln j$. Hence,

$$e^{x+yj} = j = e^{(\pi/2+2n\pi)j}, \qquad \text{for } n = 0, \pm1, \pm2, \cdots.$$

And so,

$$\ln(j) = (\tfrac{1}{2}\pi + 2n\pi)j \qquad n = 0, \pm1, \pm2, \cdots.$$

Note that we can use the same methodology to show that

$$\ln(1) = 2n\pi j \qquad n = 0, \pm1, \pm2, \cdots,$$

where the $n = 0$ case is the one that we are used to, namely that $\ln(1) = 0$.

(c) j^j.

Given that $j = e^{(\frac{1}{2}\pi + 2n\pi)j}$, we get

$$\begin{aligned} j^j &= \left[e^{(\frac{1}{2}\pi+2n\pi)j}\right]^j, \\ &= e^{(\frac{1}{2}\pi+2n\pi)j^2}, \\ &= e^{-(\frac{1}{2}\pi+2n\pi)}, \end{aligned}$$

for $n = 0, \pm 1, \pm 2, \cdots$. Therefore j^j yields an infinite set of real values. Strange......

The same idea also gives,

$$1^j = e^{-2n\pi} \qquad \text{for } n = 0, \pm 1, \pm 2, \cdots,$$

which is clearly completely crackers.

(d) $y = \cos^{-1} 2$.

We may take cosines of both sides to get $\cos y = 2$. Given that $\cos y = \frac{1}{2}(e^{jy} + e^{-jy})$ we have,

$$e^{jy} + e^{-jy} = 4.$$

Multiplying both sides by e^{jy} and rearranging yields a quadratic for e^{jy}:

$$e^{2jy} - 4e^{jy} + 1 = 0.$$

By applying the standard formula for the solution of a quadratic (or by completion of the square) we get,

$$e^{jy} = 2 \pm \sqrt{3}.$$

Now let $y = a + bj$ and hence,

$$e^{j(a+bj)} = e^{-b}e^{ja} = (2 \pm \sqrt{3}). \tag{1}$$

Given our experience with the roots of complex numbers, we may replace equation (1) by,

$$e^{-b}e^{ja} = (2 \pm \sqrt{3})e^{2n\pi j}$$

for integer values of n. Hence $a = 2n\pi$, and $e^{-b} = 2 \pm \sqrt{3}$. The plus/minus here is rather strange, but lets run with it for a moment to see how this resolves itself.

Now this is not an obvious step, but

$$\frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}, \tag{2}$$

which may be proved by multiplying both the numerator and the denominator by $2 + \sqrt{3}$. The above expression for b is now,

$$\begin{aligned} b &= -\ln(2 \pm \sqrt{3}) \\ &= -\ln(2 + \sqrt{3}),\ -\ln(2 - \sqrt{3}) \\ &= -\ln(2 + \sqrt{3}),\ +\ln\left(\tfrac{1}{2-\sqrt{3}}\right) \\ &= -\ln(2 + \sqrt{3}),\ +\ln(2 + \sqrt{3}) \qquad \text{using (2)} \\ &= \pm \ln(2 + \sqrt{3}). \end{aligned}$$

Hence

$$y = a + bj = 2n\pi \pm j\ln(2+\sqrt{3})$$

for $n = 0, \pm 1, \pm 2, \pm 3...$ is the final answer.

This feels rather strange so perhaps the solution needs to be checked. Therefore

$$\cos y = \cos\Big(2n\pi \pm j\ln(2+\sqrt{3})\Big) = \cos\Big(\pm j\ln(2+\sqrt{3})\Big), \tag{3}$$

after using the formula for the cosine of a sum of terms.

Given the definition of the cosine in terms of complex exponentials, both the signs in the $\pm$ in (3) yield exactly the same cosine, and therefore we shall select just one of them, the plus. Therefore,

$$\begin{aligned}
\cos y &= \cos\Big(j\ln(2+\sqrt{3})\Big) \\
&= \tfrac{1}{2}\Big(e^{j^2\ln(2+\sqrt{3})} + e^{-j^2\ln(2+\sqrt{3})}\Big) \\
&= \tfrac{1}{2}\Big(e^{\ln(2+\sqrt{3})} + e^{-\ln(2+\sqrt{3})}\Big) \\
&= \tfrac{1}{2}\Big(e^{\ln(2+\sqrt{3})} + e^{\ln[(2+\sqrt{3})^{-1}]}\Big) \\
&= \tfrac{1}{2}\Big((2+\sqrt{3}) + (2+\sqrt{3})^{-1}\Big) \\
&= \tfrac{1}{2}\Big((2+\sqrt{3}) + (2-\sqrt{3})\Big) \quad \text{using (2)} \\
&= 2.
\end{aligned}$$

Q13. Definitely a maths department type of question, but let's fill up space. A Pythagorean triple is a set of three integers which satisfy, $n^2 + m^2 = q^2$. Now if $a^2 + b^2$ is such a square, as is $c^2 + d^2$, then the product, $(a^2 + b^2)(c^2 + d^2)$ may be written as the sum of the two squares, $u^2 + v^2$, in two different ways. I've only just recently discovered this result and my initial reaction was disbelief! Your task is to prove it but I'll give a hint or two.

One may factorise $(a^2 + b^2)$ into complex factors (think complex conjugates). This means that $(a^2+b^2)(c^2+d^2)$ may be split into four complex factors. Now there are two different ways of pairing up these factors before they are multiplied. The rest is up to you.... Test out your result using

$$(3^2 + 4^2)(5^2 + 12^2).$$

A13. The hint goes back to the fact that, when $z = x + yj$ and $\bar{z} = x - yj$, then

$$z\bar{z} = x^2 + y^2,$$

and therefore we may perform the factorisation,

$$x^2 + y^2 = (x + yj)(x - yj).$$

Therefore we have,

$$(a^2 + b^2)(c^2 + d^2) = (a + bj)(a - bj)(c + dj)(c - dj).$$

If we choose to multiply the 1st and the 3rd factors, and to multiply the 2nd and 4th, then we will obtain,

$$\begin{aligned}
(a^2 + b^2)(c^2 + d^2) &= (a + bj)(c + dj) \times (a - bj)(c - dj) \\
&= \Big[(ac - bd) + (bc + ad)j\Big] \times \Big[(ac - bd) - (bc + ad)j\Big] \\
&= (ac - bd)^2 + (bc + ad)^2,
\end{aligned}$$

and this is one sum of squares.

If we now choose to multiply the 1st and the 4th factors, and to multiply the 2nd and 3rd, then we will obtain,

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= (a+bj)(c-dj)\times(a-bj)(c+dj)\\ &= \Big[(ac+bd)+(bc-ad)j\Big]\times\Big[(ac+bd)-(bc-ad)j\Big]\\ &= (ac+bd)^2+(bc-ad)^2,\end{aligned}$$

which is a different sum of squares. Absolutely astonishing.

So the result is that

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= (ac-bd)^2+(bc+ad)^2\\ &= (ac+bd)^2+(bc-ad)^2.\end{aligned}$$

If we use the numerical data given in the question, then we may set $a=3$, $b=4$, $c=5$ and $d=12$. This yields the following,

$$(3^2+4^2)(5^2+12^2)=33^2+56^2=63^2+16^2=4225.$$
