Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem Sheet 2 — Differentiation

1. Use the 'small increment' method (i.e. the approach using limits) described in the lecture notes to find the derivative of the following functions: [NOTE: **(i)** this type of question will *not* appear in the exam; **(ii)** part (d) begs the question – you'll see what I mean; **(iii)** there's a sneaky trick that you'll need to find for part (e) which is related in some way to finding z^{-1} in complex numbers.]

(a) x^3, (b) x^4, (c) x^{-1}, (d) $\sin x$, (e) $x^{1/2}$.

A1

There will not be a question on this type of analysis in the exam. Its aim is the use of infinitesimals to obtain derivatives, something which was anathema to scientists in and just before the time of Newton and Leibnitz.

(a) Using the limiting definition, we have

$$\begin{aligned}
\frac{d(x^3)}{dx} &= \lim_{\delta x\to 0}\left[\frac{(x+\delta x)^3 - x^3}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{[x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3] - x^3}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[3x^2 + 3x(\delta x) + (\delta x)^2\right] \\
&= 3x^2.
\end{aligned}$$

(b) Using the limiting definition, we have

$$\begin{aligned}
\frac{d(x^4)}{dx} &= \lim_{\delta x\to 0}\left[\frac{(x+\delta x)^4 - x^4}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{[x^4 + 4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4] - x^4}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[4x^3 + 6x^2(\delta x) + 4x(\delta x)^2 + (\delta x)^3\right] \\
&= 4x^3.
\end{aligned}$$

(c) Using the limiting definition, we have

$$\begin{aligned}
\frac{d(x^{-1})}{dx} &= \lim_{\delta x\to 0}\left[\left(\frac{1}{x+\delta x}-\frac{1}{x}\right)/\delta x\right] \\
&= \lim_{\delta x\to 0}\left[\left(\frac{x}{x(x+\delta x)}-\frac{x+\delta x}{x(x+\delta x)}\right)/\delta x\right] \\
&= \lim_{\delta x\to 0}\left[\left(\frac{x-(x+\delta x)}{x(x+\delta x)}\right)/\delta x\right] \\
&= \lim_{\delta x\to 0}\left[\left(\frac{-\delta x}{x(x+\delta x)}\right)/\delta x\right] \\
&= \lim_{\delta x\to 0}\left[-\frac{1}{x(x+\delta x)}\right] \\
&= -\frac{1}{x^2}.
\end{aligned}$$

(d) We have

$$\begin{aligned}
\frac{d\sin x}{dx} &= \lim_{\delta x\to 0}\left[\frac{\sin(x+\delta x)-\sin(x)}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{\sin(x)\cos(\delta x)+\cos(x)\sin(\delta x)-\sin(x)}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{\sin(x)[\cos(\delta x)-1]}{\delta x}+\frac{\cos(x)\sin(\delta x)}{\delta x}\right] \\
&= \sin(x)\lim_{\delta x\to 0}\left[\frac{\cos(\delta x)-1}{\delta x}\right]+\cos(x)\lim_{\delta x\to 0}\left[\frac{\sin(\delta x)}{\delta x}\right]
\end{aligned}$$

Now both of these limits may be found using l'Hôpital's rule, but this assumes the results which we are trying to prove. Alternatively, both limits may be found using Taylor's series, but again, the derivatives of both sine and cosine have to be used to do that. The only alternative that we are left with is to let δx take small values such as 10^{-3} and 10^{-4} and so on so that we can see the trend. This eventually yields $\cos x$ as our answer because the first limit tends to zero while the second tends to $1.$ This feels unsatisfactory, and that is because it is.

The "begging of the question" refers to the fact that we are assuming what we are attempting to prove, a circular argument. That said, there is a geometrical way of proving this result.

(e) We have

$$\begin{aligned}
\frac{dx^{1/2}}{dx} &= \lim_{\delta x\to 0}\left[\frac{(x+\delta x)^{1/2}-x^{1/2}}{\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{[(x+\delta x)^{1/2}-x^{1/2}]}{dx}\frac{[(x+\delta x)^{1/2}+x^{1/2}]}{[(x+\delta x)^{1/2}+x^{1/2}]}\right] \qquad \text{...dirty trick} \\
&= \lim_{\delta x\to 0}\left[\frac{(x+\delta x)-x}{[(x+\delta x)^{1/2}+x^{1/2}]\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{\delta x}{[(x+\delta x)^{1/2}+x^{1/2}]\delta x}\right] \\
&= \lim_{\delta x\to 0}\left[\frac{1}{(x+\delta x)^{1/2}+x^{1/2}}\right] \\
&= \frac{1}{2x^{1/2}}.
\end{aligned}$$

2. Find the derivatives of the following functions with respect to x:

(a) $4\sin x + 2x$, (b) $4e^{2x} + 5x^{-1}$, (c) $(bx)^{-1}$, (d) $-4 - 5x^{-2}$, (e) e^{3x-4}, (f) $\ln|2x^3|$, (g) $|x|$, (h) $\sin|x|$.

A2. (a) $4\cos x + 2$, (b) $8e^{2x} - 5x^{-2}$, (c) $-b^{-1}x^{-2}$, (d) $10x^{-3}$, (e) $3e^{3x-4}$, (f) $3x^{-1}$.

In Part (c) you need to be careful about the constant, b: we have $(bx)^{-1} = b^{-1}x^{-1}$.

Part (f) was a bit of a trick question. Note that $\ln|2x^3| = \ln|x|^3 + \ln 2 = 3\ln|x| + \ln 2$.

(g) If $y = |x|$, then $y = x$ when $x > 0$ and $y = -x$ when $x < 0$. Hence the derivatives are $+1$ and -1 in the two respective regions. Therefore one may write the solution down in either of the following closed forms:

$$y' = |x|/x \quad \text{or} \quad y' = \text{sign}(x).$$

(h) If $y = \sin|x|$ then we may play the same trick as in part (g), namely that,

$$y = \sin x \quad (x > 0), \qquad \text{and} \qquad y = \sin(-x) = -\sin x \quad (x < 0).$$

Hence

$$y' = \cos x \quad (x > 0), \qquad \text{and} \qquad y' = -\cos x \quad (x < 0).$$

In compact form we have $y' = \text{sign}\, x \cos x$.

3. Find the derivatives of the following functions with respect to t:

(a) $t\sin t$, (b) $t^{-2}e^{3t}$, (c) $t\ln t - t$, (d) $te^{-t}\cos 2t$, (e) $\sin 2t \sinh 3t$, (f) $|t|\,\sin|t|$.

A3. (a) $t\cos t + \sin t$, (b) $-2t^{-3}e^{3t} + 3t^{-2}e^{3t} = e^{3t}(3t^{-2} - 2t^{-3})$, (c) $\ln|t|$,

(d) $e^{-t}\Big[(1-t)\cos 2t - 2t\sin 2t\Big]$, (e) $2\cos 2t\,\sinh 3t + 3\sin 2t\,\cosh 3t$.

(f) This one needs a bit more thought, but we may use some results from Q2, namely that

$$\frac{d}{dt}|t| = \text{sign}\, t \quad \text{and} \quad \frac{d}{dt}\sin|t| = \text{sign}\, t\cos t.$$

Using the product rule as usual we have,

$$\begin{aligned} \frac{d}{dt}|t|\ \sin|t| &= \frac{d|t|}{dt}\sin|t| + |t|\frac{d\sin|t|}{dt} \\ &= \text{sign}\, t\sin|t| + |t|\text{sign}\, t\cos t \quad \text{using results in Q2} \\ &= \sin t + t\cos t. \end{aligned}$$

This last line has been simplified suddenly and all the modulus signs have been dropped; how come? In the first instance it is obvious that $\text{sign}\, t\sin|t| = \sin t$ when $t > 0$. When $t < 0$ we have $\text{sign}\, t\sin|t| = (-1)\sin(-t) = \sin t$. For the second term I have used $|t|\text{sign}\, t = t$ — this is one of those results which is either completely obscure or completely obvious, and the transition from complete obscurity to complete understanding will be immediate!

4. Differentiate the following problems with respect to t:

(a) e^{t^2}, (b) $\sqrt{1+t^2}$, (c) $(1+\sqrt{t})^2$, (d) $\sin[\sin(\sin t)]$, (e) $\tan(t^{1/2})$, (f) $e^{-\sin t^2}$, (g) $(\sin t)^{1/2}$, (h) $|t|^b$.

A4. These are chain rule problems. The answers are

(a) $e^{t^2} \longrightarrow 2te^{t^2}$.

(b) $\sqrt{1+t^2} \longrightarrow \dfrac{2t}{2\sqrt{1+t^2}} = \dfrac{t}{\sqrt{1+t^2}}$.

(c) $(1+\sqrt{t})^2 \longrightarrow \Big[2(1+\sqrt{t})\Big] \times \Big[t^{-1/2}/2\Big] = \dfrac{1+\sqrt{t}}{\sqrt{t}} = 1 + t^{-1/2}$.

(d) $\sin[\sin(\sin t)] \longrightarrow \Big[\cos t\Big]\Big[\cos(\sin t)\Big]\Big[\cos[\sin(\sin t)]\Big]$.

(e) $\tan(t^{1/2}) \longrightarrow \dfrac{\sec^2(t^{1/2})}{2t^{1/2}}$.

(f) $e^{-\sin t^2} \longrightarrow -2t\ \cos(t^2)\, e^{-\sin t^2}$.

(g) $(\sin t)^{1/2} \longrightarrow \dfrac{\cos t}{2(\sin t)^{1/2}}$.

In Part (c) we could have also expanded the brackets prior to differentiating.

(h) We have $y = |x|^b = v^b$ where $v = |x|$. Hence

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = bv^{b-1}\text{sign}\, x = b|x|^{b-1}\text{sign}\, x.$$

An alternative version of this may also be written down using the definition, $\text{sign}\, x = x/|x|$. Hence we have,

$$\frac{dy}{dx} = bx|x|^{b-2}.$$

We'll take a few examples with integer values of b and simplify the results. So if $b = 4$ then we have $y' = 4x|x|^2 = 4x^3$. If $b = 5$ then $y' = 5x|x|^3 = 5x^3|x|$.

5. Using the quotient rule, differentiate the following with respect to x:

(a) $\tan(ax)$, (b) $\tanh(ax)$, (c) cosec (ax), (d) $e^x/(1+x)$, (e) e^{3x}/x^2, (f) $x/(1+x^2)$.

Use $\tan = \sin / \cos$, $\tanh = \sinh / \cosh$ and cosec $= 1/\sin$.

A5. We need the quotient rule for all of these.

(a) $\dfrac{d}{dx}\Big(\dfrac{\sin ax}{\cos ax}\Big) = \dfrac{(\cos ax)(a\cos ax) - (\sin ax)(-a\sin ax)}{\cos^2 ax} = \dfrac{a(\cos^2 ax + \sin^2 ax)}{\cos^2 ax} = \dfrac{a}{\cos^2 ax}.$

This may also be written as $a\sec^2 ax.$

(b) $\dfrac{d}{dx}\Big(\dfrac{\sinh ax}{\cosh ax}\Big) = \dfrac{(\cosh ax)(a\cosh ax) - (\sinh ax)(a\sinh ax)}{\cosh^2 ax} = \dfrac{a(\cosh^2 ax - \sinh^2 ax)}{\cosh^2 ax}.$

This may be simplified to either $\dfrac{a}{\cosh^2 ax}$ or $= a\text{sech}^2 ax.$

(c) $\dfrac{d}{dx}\Big(\dfrac{1}{\sin(ax)}\Big) = -\dfrac{a\cos ax}{\sin^2 ax} = -a\cot ax$ cosec ax.

(d) $\dfrac{d[e^x/(1+x)]}{dx} = e^x\Big[\dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}\Big].$

(e) $\dfrac{d[e^3x/x^2]}{dx} = e^{3x}[3x^{-2} - 2x^{-3}] = \Big(\dfrac{3x-2}{x^3}\Big)e^{3x}.$

(f) $\dfrac{d}{dx}\Big(\dfrac{x}{1+x^2}\Big) = \dfrac{(1+x^2)(1) - (x)(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}.$

Note the similarity between the solutions of Parts (a) and (b).

6. Find an expression for dy/dx in the following cases:

(a) $y^2 + y = x$, (b) $\sin(xy) = x$, (c) $\ln|y| = y - \cos x.$

A6. (a) $\dfrac{dy}{dx}(2y+1) = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2y+1}.$

That is the simple and straightforward solution, but we may do better for this question. There are two ways to find a solution solely in terms of x. We may solve for y from the original equation; this gives

$$y = \frac{-1 \pm \sqrt{1+4x}}{2}.$$

Hence

$$y' = \pm\frac{1}{(1+4x)^{1/2}}.$$

The 'plus' in this answer corresonds to having the same plus in the expression for y. Likewise the two minuses. This is the direct route. The second is when we substitute for the y we have found into the original expression for the derivative, y'. There isn't too much difference between the two routes.

(b) $\Big[x\dfrac{dy}{dx} + y\Big]\cos xy = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \Big[\dfrac{1}{\cos xy} - y\Big]/x.$

Again, this is the straightforward solution, but there are also two routes to a solution which is in terms of x only. The first involves rearranging the original equation where y is given explicitly in terms of x:

$$y = \frac{1}{x}\sin^{-1} x.$$

Differentiation using the product rule gives,

$$y' = \frac{1}{x\sqrt{1-x^2}} - \frac{1}{x^2}\sin^{-1} x.$$

Here we have assumed that the derivative of $\sin^{-1} x = 1/\sqrt{1-x^2}$.

The second route substitutes for both y and $\cos xy = \sqrt{1-\sin^2 xy} = \sqrt{1-x^2}$ into the initial expression for the derivative.

(c) $\dfrac{dy}{dx}(y^{-1}-1) = \sin x \qquad \Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{y\sin x}{1-y}$.

7. Find the derivatives with respect to x of the following functions. You will need to use more than one of the above rules in some cases. Part (j) is rather lengthy.

(a) $\sin^{-1}(ax+b)$ (hint: let $y = \sin^{-1}(ax+b)$, find x in terms of y and then differentiate),

(b) $\sin^{-1}(\sin 2x)$, (c) $e^{x\sin x}$, (d) $(\sin x)e^{x^2}$,

(e) 2^x (hint: first show that $2^x = e^{x\ln 2}$), (f) x^x, (g) $\log_{10}|x|$,

(h) $\sinh^{-1}(ax+b)$, [N.b. this is an inverse sinh, not a reciprocal]

(i) $xe^{(x/\sqrt{1+x^2})}$, (j) $\sin(x^2)e^{x\sin x}/(1+x^2)$.

A7. (a) Let $y = \sin^{-1}(ax+b) \qquad \Rightarrow \qquad \sin y = ax+b.$

$$\Rightarrow \qquad \frac{d(\sin y)}{dx} = \frac{d(\sin y)}{dy}\frac{dy}{dx} = (\cos y)\frac{dy}{dx} = a$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{a}{\cos y} = \frac{a}{\sqrt{1-\sin^2 y}} = \frac{a}{\sqrt{1-(ax+b)^2}}.$$

(b) Using the same trick as in part (a) we have,

$$\sin y = \sin 2x \quad \Rightarrow \quad \frac{dy}{dx}\cos y = 2\cos 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2\cos 2x}{\sqrt{1-\sin^2 y}} = \frac{2\cos 2x}{\sqrt{1-\sin^2 2x}} = \frac{2\cos 2x}{\cos 2x} = 2.$$

Perhaps this is not surprising for $\sin y = \sin 2x \Rightarrow y = 2x + 2n\pi$ where n is an integer.

(c) Answer is $\Big[x\cos x + \sin x\Big]e^{x\sin x}$.

(d) Answer is $\Big[\cos x + 2x\sin x\Big]e^{x^2}$.

(e) If we set $y = 2^x$, then $\ln y = x\ln 2$, and hence $y = e^{x\ln 2}$, as required. Hence $y' = (\ln 2)e^{x\ln 2} = (\ln 2)2^x$.

(f) Following the same procedure as in part (e), we set $y = x^x$. Then $\ln y = x\ln x$ and hence $y = e^{x\ln x}$ [Note that I am assuming the $x > 0$ in this question]. Therefore

$$y' = (1+\ln x)e^{x\ln x} = (1+\ln x)x^x.$$

(g) If $y = \log_{10}|x|$, then $10^y = |x|$. Taking natural logarithms results in,

$$y \ln 10 = \ln|x|,$$

and therefore $y' = 1/(x \ln 10)$.

(h)

$$\text{Let} \qquad y = \sinh^{-1}(ax+b) \qquad \Rightarrow \qquad \sinh y = ax+b$$

$$\Rightarrow \qquad \frac{d(\sinh y)}{dx} = \frac{d(\sinh y)}{dy}\frac{dy}{dx} = (\cosh y)\frac{dy}{dx} = a$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{a}{\cosh y} = \frac{a}{\sqrt{1+\sinh^2 y}} = \frac{a}{\sqrt{1+(ax+b)^2}}.$$

(i)

$$y = xe^{(x/\sqrt{x^2+1})}.$$

Ho, a product where one function is a function of a quotient where the denominator is a function of a function! So we'll check that quotient first.

So the derivative of $\sqrt{x^2+1}$ is $x/\sqrt{x^2+1}$.

Then the derivative of $x/\sqrt{x^2+1}$ is,

$$\frac{\sqrt{x^2+1}\times 1 - x\times(x/\sqrt{x^2+1})}{x^2+1} = \frac{1}{(x^2+1)^{3/2}}$$

after simplification.

Hence the derivative of $e^{x/\sqrt{x^2+1}}$ is $\dfrac{e^{x/\sqrt{x^2+1}}}{(x^2+1)^{3/2}}$.

Finally, we may say that,

$$\frac{dy}{dx} = e^{x/\sqrt{x^2+1}} \times \left[1 + \frac{x}{(x^2+1)^{3/2}}\right].$$

(j) Answer is

$$e^{x\sin x}\left[\frac{2x\cos x^2}{1+x^2} + \frac{(x\cos x + \sin x)\sin x^2}{1+x^2} - \frac{2x\sin x^2}{(1+x^2)^2}\right].$$

This one is a bit of a tour-de-force. Because this is so ridiculously complicated and so unsuitable for an exam, I'll leave the checking to you — sorry!

8. **[Lengthy and challenging.]** Find the first, second and third derivatives of $x^n e^{ax}$, where we can assume that $n > 3$. Can you write down a compact expression for the m^{th} derivative of this function?

A8. If we set $y = x^n e^{ax}$, then one application of the product rule gives,

$$\begin{aligned} y' &= nx^{n-1}e^{ax} + a\,x^n e^{ax} \\ &= x^{n-1}e^{ax}\Big[n + ax\Big]. \end{aligned}$$

The second derivative may be found by differentiating the first line of the above expression for y',

$$\begin{aligned} y'' &= n(n-1)x^{n-2}\,e^{ax} + 2nax^{n-1}\,e^{ax} + a^2\,x^n\,e^{ax} \\ &= x^{n-2}\,e^{ax}\Big[n(n-1) + 2nax + a^2x^2\Big]. \end{aligned}$$

The third derivative, after some tidying up gives,

$$y''' = x^{n-3}\,e^{ax}\Big[n(n-1)(n-2) + 3n(n-1)ax + 3na^2x^2 + a^3x^3\Big].$$

This may be tidied up further:

$$y''' = n!\,x^{n-3}\,e^{ax}\Big[\frac{1}{(n-3)!} + 3\frac{(ax)}{(n-2)!} + 3\frac{(ax)^2}{(n-1)!} + \frac{(ax)^3}{n!}\Big].$$

Now that last manoeuvre was unexpected but its effect in huge in terms of simplifying the resulting expressions.

Noting now that the coefficients of the quotients are the binomial coefficients, which we shall denote and define as,

$$^{p}C_q = \binom{p}{q} = \frac{p!}{q!(p-q)!},$$

we may rewrite this formula for y''' in the form,

$$\begin{aligned} y''' &= n!\,x^{n-3}\,e^{ax}\Big[\frac{1}{(n-3)!}\binom{3}{0} + \frac{(ax)}{(n-2)!}\binom{3}{1} + \frac{(ax)^2}{(n-1)!}\binom{3}{2} + \frac{(ax)^3}{n!}\binom{3}{3}\Big] \\ &= n!\,x^{n-3}\,e^{ax}\textstyle\sum_{i=0}^{3}\displaystyle\frac{(ax)^i}{(n-3+i)!}\binom{3}{i}. \end{aligned}$$

Therefore the m^{th} derivative is

$$y^{(m)} = n!\,x^{n-m}\,e^{ax}\sum_{i=0}^{m}\frac{(ax)^i}{(n-m+i)!}\binom{m}{i}.$$

Strictly speaking, this formula is perfectly correct when $n \geq m$; this is when the power of x in the original expresion for y is greater than or equal to the number of differentiations. When $n < m$, then all the terms where the exponent of x is negative need to be discarded. That this should be so is not obvious from our final formula, but if you look at the first expression for y''' above, for which $m = 3$, then see what happens to the coefficients if we choose $n = 2$....

Again, this type of question is beyond what I expect in the examination.

9. **[Challenging.]** This one is the strangest one.... By considering a simple sketch it is easy to be convinced of the fact that $dy/dx = 1/(dx/dy)$. Use this result and the chain rule to find the appropriate formula for d^2y/dx^2 in terms of d^2x/dy^2. Check that your final formula is correct by applying it to $y = \ln x$ (for $x > 0$) and to $y = x^2$.

A9. We are given that

$$\frac{dy}{dx} = 1\Big/\frac{dx}{dy}.$$

This is reasonable because a slope is one increment divided by the other, and this expression reflects the fact that one can use the reciprocal of this slope in the limiting process to find the derivative of x with respect to y.

Now we will use the chain rule as follows.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right] \\ &= \frac{dy}{dx} \times \frac{d}{dy}\left[\frac{dy}{dx}\right] && \text{chain rule} \\ &= \frac{dy}{dx} \times \frac{d}{dy}\left[\frac{dx}{dy}\right]^{-1} && \text{using given result} \\ &= \frac{dy}{dx} \times \frac{\dfrac{d^2x}{dy^2}}{-\left(\dfrac{dx}{dy}\right)^2} && \text{diff. with respect to } y \\ &= -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3} && \text{using given result.}\end{aligned}$$

Note that this formula may be rearranged to give,

$$\frac{d^2x}{dy^2} = -\frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}. \qquad (1)$$

The first case for checking is,

$$y = \ln x \quad \Rightarrow \quad \frac{dy}{dx} = x^{-1} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -x^{-2}.$$

If $y = \ln x$ then $x = e^y$. Therefore we have

$$x = e^y \quad \Rightarrow \quad \frac{dx}{dy} = e^y \quad \Rightarrow \quad \frac{d^2x}{dy^2} = e^y.$$

The right hand side of the formula in Eq. (1) gives,

$$\text{RHS} = -\frac{-x^{-2}}{(x^{-1})^3} = x = e^y = \text{LHS},$$

i.e. the left hand side of (1). So the formula works.

In the second case we have,

$$y = x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 2.$$

We also have,

$$x = y^{1/2} \quad \Rightarrow \quad \frac{dx}{dy} = \tfrac{1}{2}y^{-1/2} \quad \Rightarrow \quad \frac{d^2x}{dy^2} = -\tfrac{1}{4}y^{-3/2}.$$

So the RHS of Eq. (1) gives,

$$\text{RHS} = -\frac{2}{(2x)^3} = -\tfrac{1}{4}x^{-3} = -\tfrac{1}{4}y^{-3/2} = \text{LHS}.$$