Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem Sheet 3 — Differentiation

1. Find the critical points of the following functions. Which are maxima and which are minima? Find the values of the functions at these points and sketch the functions.

(a) $f(x) = 2x^2 - 3$,

(b) $g(t) = t^2(t^2 - 1)$,

(c) $h(y) = ye^{-y}$,

(d) $F(x) = x^2 \sin^2 x$,

(e) $G(x) = x^4 + 2x^3 - 2x - 1$,

(f) $H(x) = x^2 e^{-x^2}$,

(g) $\theta(z) = 3z^4 - 2z^6$,

(h) $\Phi(\alpha) = (\alpha^2 - 4\alpha - 20)e^{-\alpha}$.

A1. (a) $f(x) = 2x^2 - 3$,

Here $f'(x) = 4x$ and $f''(x) = 4$, so the only critical point is at $x = 0$ where $f(0) = -3$. As $f''(0) > 0$ this is a minimum, but this fact could have been guessed from your knowledge of the behaviour of a parabola.

(b) $g(t) = t^2(t^2 - 1)$.

Here $g'(t) = 2t(2t^2 - 1)$ and $g''(t) = 12t^2 - 2$. Therefore critical points occur at $t = 0, \pm 1/\sqrt{2}$.

Hence:

$g(-1/\sqrt{2}) = -1/4$ is a local minimum as $g''(-1/\sqrt{2}) = 4 > 0$.

$g(1/\sqrt{2}) = -1/4$ is also a local minimum as $g''(1/\sqrt{2}) = 4 > 0$.

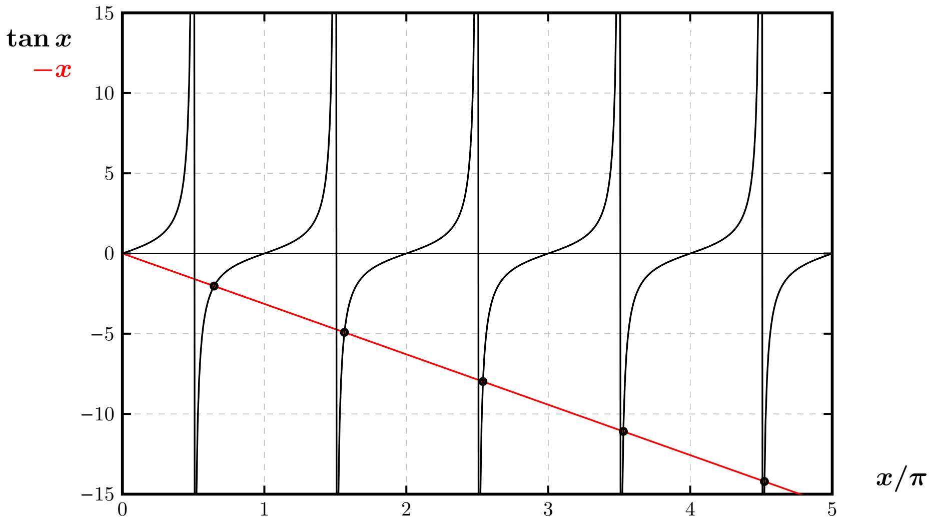
$g(0) = 0$ is a local maximum as $g''(0) = -2 < 0$.

(c) $h'(y) = (1 - y)e^{-y}$ and this is zero when $y = 1$. So $h(1) = e^{-1}$. The second derivative is $h''(y) = (y - 2)e^{-y}$ and this is negative when $y = 1$, and so $y = 1$ corresponds to a maximum.

(d) $F(x) = x^2 \sin^2 x$. **[Warning: this is a tricky solution, so don't fret about this sort of question arising in the exam.]**

Here $F'(x) = x \sin x \Big(2 \sin x + 2x \cos x\Big)$. All three factors in F' are zero when $x = 0$, so this is certainly an interesting point! In fact, given that $\sin x$ looks like x when x is small, then $F \simeq x^4$ near the origin. So on these grounds $x = 0$ is a quartic minimum, rather than the usual parabolic minimum. Strictly speaking, if we weren't to appeal to the fact that $\sin x \simeq x$ when x is small, we would have to show that $F''(0) = F'''(0) = 0$ and $F''''(0) < 0$ to show that $x = 0$ is a quartic minimum.

The other zeros of $F'(x) = 0$ arise (i) when $\sin x = 0$ (i.e. when $F(x)$ touches the x-axis — you'll need to sketch the function to see this clearly), and (ii) when $\sin x + x \cos x = 0$, which corresponds to when $\tan x = -x$). It isn't possible to solve $\tan x = -x$ analytically, but when x is large and positive, then $\tan x$ must be large and negative, which means that x must be close to, but slightly above $\pi/2 + n\pi$ where n is a large integer. This is illustrated below where the black curve is $\tan x$, the red curve is $-x$, while the black disks are where they cross and hence the x-values are the locations of the critical points of $F(x)$.



We haven't sketched the function $F(x)$ but it is always positive. Therefore the critical points at $x = \pm n\pi$, where $\sin x = 0$, must be minima. A already stated, the critical point at a$x = 0$ is a quartic minimum. Then, by process of elimination in this case, the zeros of $x + \tan x = 0$ as shown above, will be maxima. Strictly speaking one should differentiate F' and check the sign of F'' at each of these roots of F'.

(e) $G(x) = x^4 + 2x^3 - 2x - 1$.

We get $G'(x) = 4x^3 + 6x^2 - 2$. By inspection, I see that $x = -1$ is a root of $G'(x) = 0$, and therefore it is possible to factorise the cubic:

$$G'(x) = 4x^3 + 6x^2 - 2 = (x + 1)(4x^2 + 2x - 2).$$

The resulting quadratic looks a little tricky to factorise, but if we set it to be zero, then its roots turn out to be $x = -1, \frac{1}{2}$. Hence, it is $4(x + 1)(x - \frac{1}{2})$. Therefore

$$G'(x) = 4x^3 + 6x^2 - 2 = 4(x + 1)^2(x - \tfrac{1}{2}).$$

So we have two values of x at which $G' = 0$, namely $x = -1$ (twice) and $x = \frac{1}{2}$.

We may examine the nature of these extrema by determining the value of the second derivative of G at these points. We have

$$G''(x) = 12x^2 + 12x = 12x(1 + x).$$

Given that $G''(\frac{1}{2}) = 9$, then $x = \frac{1}{2}$ is a minimum. But we see that $G''(-1) = 0$, which means that our test is inconclusive so far. However,

$$G'''(x) = 24x + 12,$$

which is nonzero at $x = -1$. Therefore the curve has a point of inflexion at $x = -1$. And $G'''(-1) = -12$ tells us that it is a descending point of inflexion.

Finally, I'll note that the original function is

$$G(x) = (x + 1)^3(x - 1)$$

which shows that $G(x) \simeq -2(x + 1)^3$ when x is very close to -1, which also confirms the diagnosis of a descending point of inflexion there.

(f) $H(x) = x^2 e^{-x^2}$. We find that

$$H' = e^{-x^2}(2x - 2x^3) = 2x(1 - x^2)e^{-x^2}.$$

Hence the three critical points are at $x = 0, 1, -1$.

We need to find the second derivative. After a little tidying up, we find that,

$$H'' = e^{-x^2}\Big[2 - 10x^2 + 4x^4\Big].$$

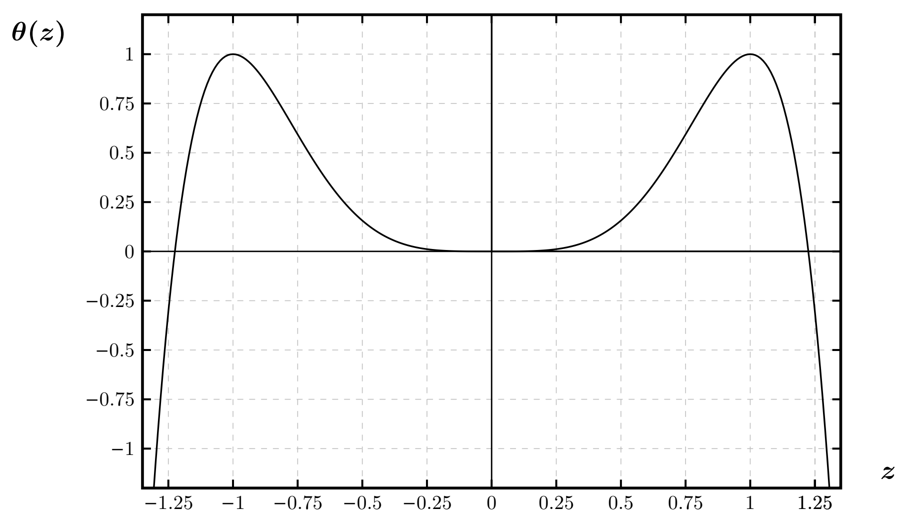
From this we find that $F''(0) = 2$ while $F''(\pm 1) = -4e^{-1}$.

Thus $x = 0$ is a minimum while $x = \pm 1$ are maxima.

(g) $\theta(z) = 3z^4 - 2z^6$,

$\theta(z) = 3z^4 - 2z^6$		
$\theta'(z) = 12z^3 - 12z^5$	$\Rightarrow z = 0, \pm 1$ are critical points	
$\theta''(z) = 36z^2 - 60z^4$	$\Rightarrow \theta''(\pm 1) = 24$	$\Rightarrow z = \pm 1$ are maxima
	$\Rightarrow \theta''(0) = 0$	$\Rightarrow z = 0$ is not a minimum nor a maximum
$\theta'''(z) = 72z - 240z^3$	$\Rightarrow \theta'''(0) = 0$	$\Rightarrow z = 0$ is not an inflexion point
$\theta''''(z) = 72 - 720z^2$	$\Rightarrow \theta''''(0) = 72$	$\Rightarrow z = 0$ is a quartic minimum

These conclusions are confirmed in the following Figure, where the quartic minimum at $z = 0$ is very clearly much flatter than a standard minimum is:



(h) $\Phi(\alpha) = (\alpha^2 - 4\alpha - 20)e^{-\alpha}$.

Hence,

$$\Phi'(\alpha) = (16 + 6\alpha - \alpha^2)e^{-\alpha} = (2 + \alpha)(8 - \alpha)e^{-\alpha}.$$

Therefore the two critical points are at $\alpha = -2$ and $\alpha = 8$.

What sort of extrema are they? We find that,

$$\Phi''(\alpha) = (\alpha^2 - 8\alpha - 10)e^{-\alpha},$$

and hence $\Phi''(-2) = 10e^2 > 0$, and $\Phi''(8) = -10e^{-8} < 0$. So we conclude that $\alpha = 8$ is a maximum and $\alpha = -2$ is a minimum.

Given that the minimum is $\Phi(-2) = -8e^2 \simeq -59$ and the maximum is $\Phi(8) = 12e^{-8} \simeq 0.00403$, I also conclude that it'll be difficult to see each of the critical points simultaneously on a graph.

2. Find and classify any critical points of the function,

$$y = 10x^6 - 36x^5 + 45x^4 - 20x^3 + 1.$$

A2. The function is $y = 10x^6 - 36x^5 + 45x^4 - 20x^3 + 1$. Hence the derivative is,

$$\begin{aligned} y' &= 60x^5 - 180x^4 + 180x^3 - 60x^2 && \text{We may now factor out } 60x^2 \\ &= 60x^2(x^3 - 3x^2 + 3x - 1) && \text{looks like } (x-1)^3 \\ &= 60x^2(x-1)^3 \end{aligned}$$

and hence $x = 0$ and $x = 1$ are critical points.

Using the second line of the above workings we have,

$$y'' = 60(5x^4 - 12x^3 + 9x^2 - 2x).$$

Hence $y''(0) = 0$ and therefore $x = 0$ is neither a minimum nor a maximum.

Also $y''(1) = 0$ and so $x = 1$ is neither a minimum nor a maximum either.

Now,

$$y''' = 60(20x^3 - 36x^2 + 18x - 2).$$

So $y'''(0) = -120$ which means that $x = 0$ is a descending inflexion point.

Also $y'''(1) = 0$ and therefore $x = 1$ isn't a point of inflexion.

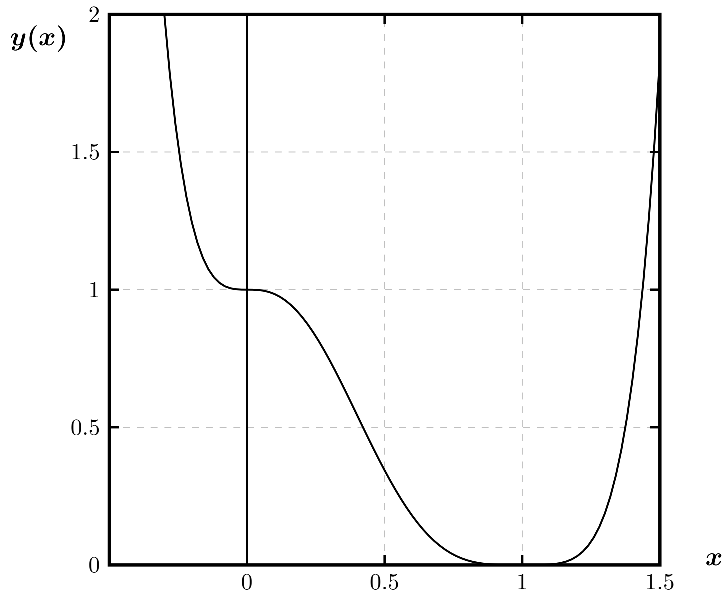
Hopefully, finally....

$$y'''' = 60(60x^2 - 72x + 18),$$

and so $y''''(1) = 360$ which means that $x = 1$ is a quartic minimum.

In summary, we have a descending inflexion point at $x = 0$ and a quartic minimum at $x = 1$.

This was perhaps the easiest example that I could think of with both a point of inflexion and a quartic extremum but which is tractable in a fairly reasonable amount of time. But what does it look like? Here it is:



3. Find and classify any critical points of the function,

$$y = e^x/(x^2+1).$$

[Hint, this becomes very very messy after the first derivative. Therefore I would suggest that one should set $y' = (x-1)^2 f(x)$ at that point before obtaining any higher derivatives.]

A3. If $y = \dfrac{e^x}{x^2+1}$ then,

$$y' = e^x\Big[\frac{1}{x^2+1} - \frac{2x}{(x^2+1)^2}\Big] = e^x\Big[\frac{x^2+1-2x}{(x^2+1)^2}\Big] = (x-1)^2\frac{e^x}{(x^2+1)^2}.$$

Using the hint, let $y'(x) = (x-1)^2 f(x)$ where $f(x) = \dfrac{e^x}{(x^2+1)^2}$. So we see that $y'(x) = 0$ only when $x = 1$.

Hence,

$$y'' = (x-1)^2 f'(x) + 2(x-1)f(x),$$

which means that $f''(1) = 0$, and therefore this point is neither a maximum nor a minimum.

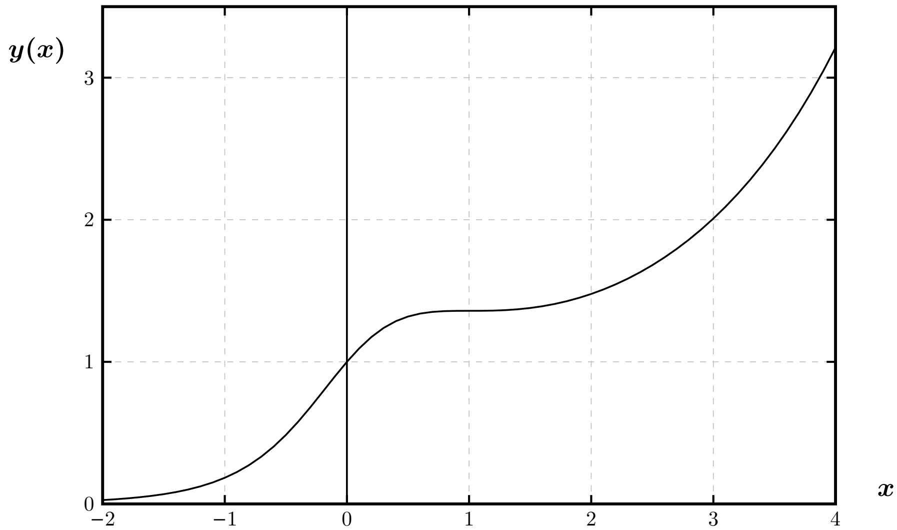
Taking another derivative gives,

$$y''' = (x-1)^2 f''(x) + 4(x-1)f'(x) + 2f(x),$$

and therefore $y'''(1) = 2f(1) = \frac{1}{4}e > 0.$ So we have a rising point of inflexion at $x = 1$.

On this occasion the use of the $f(x)$ substitution meant that we didn't have to find a very complicated set of derivative and second derivatives. I have to say that the presence of a point of inflexion here is very surprising indeed for there is no hint at all that it is likely.

Finally, the variation of y with x is shown below, and the rising inflextion point is seen easily at $x = 1$.



y(x)
x
0
1
2
3
−2
−1
0
1
2
3
4