Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem Sheet 4 — Integration

1. Find the following integrals:

(a) $\int x^3\,dx$, (b) $\int x^{-5}\,dx$, (c) $\int e^{-4t}\,dt$, (d) $\int \cosh 2y\,dy$,

(e) $\int_0^1 (2x+3)\,dx$, (f) $\int_{-1}^2 (4x^2-3x)\,dx$, (g) $\int_0^{\pi/4} (2\cos\alpha+\sin\alpha)\,d\alpha$.

A1. Just the answers here.

(a) $\int x^3\,dx = x^4/4 + c.$

(b) $\int x^{-5}\,dx = -x^{-4}/4 + c.$

(c) $\int e^{-4t}\,dt = -e^{-4t}/4 + c.$

(d) $\int \cosh 2y\,dy = \frac{1}{2}\sinh 2y + c.$

(e) $\int_0^1 (2x+3)\,dx = 4.$

(f) $\int_{-1}^2 (4x^2-3x)\,dx = \Big[\frac{4}{3}x^3 - \frac{3}{2}x^2\Big]_{-1}^2 = 7.5.$

(g) $\int_0^{\pi/4} (2\cos\alpha+\sin\alpha)\,d\alpha = \Big[2\sin\alpha - \cos\alpha\Big]_0^{\pi/4} = 1 + 1/\sqrt{2}.$

2. Find the following integrals:

(a) $\int (x+3)^{-2}\,dx$, (b) $\int_0^1 2xe^{x^2}\,dx$ (hint: substitute $z = x^2$),

(c) $\int \frac{1}{x^2+a^2}\,dx$ (hint: substitute $x = a\tan\theta$),

(d) $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ (hint: substitute $z = \cos x$),

(e) $\int \frac{x}{1+x^2}\,dx$, (f) $\int \cos x\cos(\sin x)\,dx$, (g) $\int \theta^2\sin(\theta^3)\,d\theta$, (h) $\int_0^{\pi/2} \sqrt{\sin t}\,\cos t\,dt$,

(i) $\int \Big[f(x)\Big]^n f'(x)\,dx$, (j) $\int (ff'' + f'f')\,dx$ (Try to find the integral by inspection),

(k) $\int_0^1 \sqrt{1-x^2}\,dx$, (l) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$,

(m) $\int_0^1 (1+\ln x)x^x\,dx$ [you will need to rewrite x^x in an alternative form (hint: take logs), and to assume that $\lim_{x\to 0} x^x = 1$].

A2. With some explanations this time.

(a) $\displaystyle\int (x+3)^{-2}\,dx = -(x+3)^{-1} + c,$

(b) $I = \displaystyle\int_0^1 2xe^{x^2}\,dx$ (hint: substitute $z = x^2$).

Let $z = x^2$ and therefore $dz = 2x\,dx$. The limits transform as follows: $x = 0 \implies z = 0$ and $x = 1 \implies z = 1$. So on this occasion, the limits remain the same.

Hence $I = \displaystyle\int_0^1 e^z\,dz = \Big[e^z\Big]_0^1 = e - 1.$

(c) $\displaystyle\int \frac{1}{x^2+a^2}.$

If we set $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$. Hence the integral becomes,

$$\int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}\,d\theta = \int \frac{1}{a}\,d\theta,$$

because $\sec^2\theta = 1+\tan^2\theta$. Therefore the integral becomes $\theta/a + c$, or, when written in terms of the original variable,

$$\frac{\tan^{-1}(x/a)}{a} + c.$$

(d) $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx.$

Let $z = \cos z$ and therefore $dz = -\sin x\,dx$. The limits transform as follows: $x = 0 \implies z = 1$ and $x = \pi/2 \implies z = 0$.

Hence $I = \displaystyle\int_1^0 \frac{-dz}{1+z^2} = \int_0^1 \frac{dz}{1+z^2} = \Big[\tan^{-1} z\Big]_0^1 = \frac{\pi}{4}$. Here we have used the integral determined in part (c).

(e) $I = \displaystyle\int \frac{x}{1+x^2}\,dx.$

Let $z = x^2$ and therefore $dz = 2x\,dx$. Hence

$$I = \tfrac{1}{2}\int \frac{dz}{1+z} = \tfrac{1}{2}\ln|1+z| + c = \tfrac{1}{2}\ln(1+x^2) + c.$$

Note that I haven't used the modulus signs in the final answer because $(1+x^2)$ is always positive.

(f) $\displaystyle\int \cos x\,\cos(\sin x)\,dx = \sin(\sin x) + c$ using $z = \sin x$.

(g) $\displaystyle\int \theta^2\sin(\theta^3)\,d\theta = -\tfrac{1}{3}\cos(\theta^3) + c,$ using $z = \theta^3$.

(h) $\displaystyle\int_0^{\pi/2} \sqrt{\sin t}\,\cos t\,dt = \int_0^1 z^{1/2}dz = \Big[\tfrac{2}{3}z^{3/2}\Big]_0^1 = \tfrac{2}{3}$ using $z = \sin t$.

(i) $\displaystyle\int \Big[f(x)\Big]^n f'(x)\,dx = [f(x)]^{n+1}/(n+1)$, using $y = f(x)$ as the substitution.

(j) $\displaystyle\int (ff'' + f'f')\,dx = ff' + c$. This is a 'by inspection' answer. Or...one could notice that the integrand is the sort of thing that might arise from the differentiation of a product, and then the sleuthing happens...

Note that all indefinite integrals are completed by adding on an arbitrary constant.

(k) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx.$

Given that $\sin^2 + \cos^2 = 1$, or, more conveniently for the present purpose, that $1 - \sin^2 = \cos^2$, we'll let $x = \sin\theta$, and therefore $dx = \cos\theta\,d\theta$. The limits transform as follows,

$$x = 0 \quad\Rightarrow\quad \theta = 0, \qquad\qquad x = 1 \quad\Rightarrow\quad \theta = \tfrac{1}{2}\pi.$$

Hence the integral becomes,

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \\ &= \int_0^{\pi/2} \cos^2\theta\,d\theta \\ &= \int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \tfrac{1}{4}\pi.\end{aligned}$$

The final integral may be done quickly because the $\cos 2\theta$ function is to be integrated over half of its period from $\theta = 0$, and therefore that part of the integral must be zero.

(l) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$

This integral may be transformed in the same way as the last one, including the limits. Therefore we have,

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta}\,d\theta = \int_0^{\pi/2} d\theta = \Big[\theta\Big]_0^{\pi/2} = \tfrac{1}{2}\pi.$$

(m) $\displaystyle\int_0^1 (1+\ln x)x^x\,dx.$

First, we need to change x^x as given in the hint. If we set $y = x^x$ and take natural logarithms, then, by the standard rules of logarithms, we have

$$\ln y = x\ln x.$$

If we now go 'backwards' then we get $y = e^{x\ln x}$. Therefore our integral becomes,

$$\int_0^1 (1+\ln x)e^{x\ln x}\,dx.$$

Given that the derivative of $x\ln x$ is $(1+\ln x)$, it is clear that we must use the substitution, $z = x\ln x$, and so $dz = (1+\ln x)\,dx$. With regard to the limits, when $x = 0$ then $z = 0$ [note, it is given that $x^x = e^{x\ln x}$

is equal to 1 when $x = 0$, and therefore $x \ln x$ must be equal to zero. When $x = 1$, we also find that $z = 0$. Therefore the original integral transforms to,

$$\int_0^0 e^z \, dz,$$

which must be zero because we are integrating over a zero-length range.

I agree, this is a very strange result, and is unexpected. It feels as though something has gone wrong somewhere. In view of this, I have checked the result using numerical methods, and have confirmed that the integral is indeed zero.

3. Use partial fractions to simplify the integrand in the following integrals, and hence find those integrals:

(a) $\displaystyle\int \frac{2t+3}{t^2+3t+2}\,dt$, (b) $\displaystyle\int \frac{1}{t^2+3t+2}\,dt$, (c) $\displaystyle\int \frac{1}{t^3+t}\,dt$, (d) $\displaystyle\int \frac{t+3}{t^3+3t^2+2t}\,dt$,

(e) $\displaystyle\int \frac{2t+1}{t^2(t+1)}\,dt$, (f) $\displaystyle\int \frac{t^2-2}{(t+1)(t^2+2t+2)}\,dt$, (g) $\displaystyle\int \frac{1}{t(t^2+1)^2}\,dt$.

Please note that parts (f) and (g) are quite lengthy.

A3. A reminder that repeated linear factors are treated in the standard way given in the lectures. An isolated quadratic factor is treated as a linear term divided by the quadratic factor. A repeated quadratic factor will involve both of those techniques simulateously.

(a) Let $I = \displaystyle\int \frac{2t+3}{t^2+3t+2}\,dt$. Now let

$$\frac{2t+3}{t^2+3t+2} = \frac{A}{t+2} + \frac{B}{t+1}.$$

Putting both fractions over a common denominator yields

$$\frac{2t+3}{t^2+3t+2} = \frac{A(t+1)+B(t+2)}{t^2+3t+2}$$

and so

$$A(t+1)+B(t+2) = 2t+3.$$

The setting of $t = -1$ and $t = -2$ yields $A = B = 1$. Therefore we obtain

$$I = \int \Big[\frac{1}{t+2} + \frac{1}{t+1}\Big]\,dt = \ln|t+2| + \ln|t+1| + c = \ln|t^2+3t+2| + c.$$

Note that this integral is also of the form $\int (f'/f)\,dt$, for which the solution is $\ln|f| + c$.

(b) Following the same partial fractions analysis yields $B = 1$ and $A = -1$. Therefore

$$\int \frac{1}{t^2+3t+2}\,dt = \int \Big[\frac{1}{t+1} - \frac{1}{t+2}\Big]\,dt = \ln|t+1| - \ln|t+2| + c = \ln\Big|\frac{t+1}{t+2}\Big| + c.$$

(c) The integrand may be shown to reduce to

$$\frac{1}{t^3+t} = \frac{1}{t} - \frac{t}{t^2+1},$$

and hence

$$\int \frac{1}{t^3+t}\,dt = \ln|t| - \tfrac{1}{2}\ln(1+t^2) + c = \ln\Big|\frac{t}{\sqrt{1+t^2}}\Big| + c.$$

(d) Let $I = \displaystyle\int \frac{t+3}{t^3+3t^2+2t}\,dt$.

The denominator may be factorised to $t(t+1)(t+2)$. Therefore let

$$\frac{t+3}{t^3+3t^2+2t} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t+2},$$

and hence

$$t+3 = A(t+1)(t+2) + Bt(t+2) + Ct(t+1).$$

Hence,

$$\begin{aligned} t=0 &\quad\Rightarrow\quad 3=2A \quad\Rightarrow\quad A=\tfrac{3}{2},\\ t=-1 &\quad\Rightarrow\quad 2=-B \quad\Rightarrow\quad B=-2,\\ t=-2 &\quad\Rightarrow\quad 1=2C \quad\Rightarrow\quad C=\tfrac{1}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \int\Big[\frac{3/2}{t} - \frac{2}{t+1} + \frac{1/2}{t+2}\Big]\,dt,\\ &= \tfrac{3}{2}\ln|t| - 2\ln|t+1| + \tfrac{1}{2}\ln|t+2| + c,\\ &= \ln\Big[\frac{|t|^{3/2}\ |t+2|^{1/2}}{|t+1|^2}\Big] + c. \end{aligned}$$

(e) Let $I = \displaystyle\int \frac{2t+1}{t^2(t+1)}\,dt.$

Let

$$\frac{2t+1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1},$$

where we have adopted the form which is appropriate for repeated factors.

Hence,

$$2t+1 = At(t+1) + B(t+1) + Ct^2.$$

From this we take the following route:

$$\begin{aligned} t=0 &\quad\Rightarrow\quad 1=B,\\ t=-1 &\quad\Rightarrow\quad -1=C,\\ \text{coeff. of } t^2 &\quad\Rightarrow\quad 0=A+C \quad\Rightarrow\quad A=1. \end{aligned}$$

Hence

$$\begin{aligned} I &= \int \frac{2t+1}{t^2(t+1)}\,dt\\ &= \int\Big[\frac{1}{t} + \frac{1}{t^2} - \frac{1}{t+1}\Big]\,dt\\ &= \ln|t| - t^{-1} - \ln|t+1| + c\\ &= \ln\Big|\frac{t}{t+1}\Big| - t^{-1} + c. \end{aligned}$$

(f) Let $I = \int \frac{t^2 - 2}{(t+1)(t^2+2t+2)}\,dt.$

We may now set,

$$\frac{t^2 - 2}{(t+1)(t^2+2t+2)} = \frac{A}{t+1} + \frac{B+Ct}{t^2+2t+2},$$

because the quadratic factor does not have real linear factors. Hence

$$t^2 - 2 = A(t^2+2t+2) + (B+Ct)(t+1).$$

From this we take the following route:

$$\begin{array}{rcl} t = -1 & \Rightarrow & -1 = A \\ t = 0 & \Rightarrow & -2 = 2A + B \quad \Rightarrow \quad B = 0 \\ \text{coeff. of } t^2 & \Rightarrow & 1 = A + C \quad \Rightarrow \quad C = 2. \end{array}$$

Hence,

$$\begin{aligned} I &= \int \frac{t^2 - 2}{(t+1)(t^2+2t+2)}\,dt \\ &= \int \Big[-\frac{1}{t+1} + \frac{2t}{t^2+2t+2}\Big]\,dt \\ &= \int \Big[-\frac{1}{t+1} + \frac{2t+2}{t^2+2t+2} - \frac{2}{t^2+2t+2}\Big]\,dt && \text{2nd term of form } f'/f \\ &= \int \Big[-\frac{1}{t+1} + \frac{2t+2}{t^2+2t+2} - \frac{2}{(t+1)^2+1}\Big]\,dt && \text{3rd term in } \tan^{-1} \text{ form} \\ &= -\ln|t+1| + \ln|t^2+2t+2| - 2\tan^{-1}(t+1) + c. \end{aligned}$$

(g) Let $I = \int \frac{1}{t(t^2+1)^2}\,dt.$

On this occasion we have a repeated irreducible quadratic factor. Hence we set

$$\frac{1}{t(t^2+1)^2} = \frac{A}{t} + \frac{B+Ct}{t^2+1} + \frac{D+Et}{(t^2+1)^2},$$

and so

$$1 = A(t^2+1)^2 + (B+Ct)t(t^2+1) + (D+Et)t,$$

or

$$1 = A(t^4+2t^2+1) + (B+Ct)(t^3+t) + (D+Et)t.$$

It is probably best just to compare the coefficients of the various powers of t:

$$\begin{array}{rl} t^4: & A + C = 0, \\ t^3: & B = 0, \\ t^2: & 2A + C + E = 0, \\ t^1: & B + D = 0, \\ 1: & A = 1, \end{array}$$

and so $A = 1$, $B = 0$, $C = -1$, $D = 0$ and $E = -1$. Therefore the integral becomes,

$$\begin{aligned} I &= \int \frac{1}{t(t^2+1)^2}\, dt \\ &= \int \Big[\frac{1}{t} - \frac{t}{t^2+1} - \frac{t}{(t^2+1)^2}\Big]\, dt \\ &= \int \Big[\frac{1}{t} - \frac{1}{2} \times \frac{2t}{t^2+1} - \frac{t}{(t^2+1)^2}\Big]\, dt. \end{aligned}$$

The first two integrals are of the form f'/f and can be written down easily. The third may be dealt with by substitution. We let $y = t^2 + 1$ and so $dy = 2t\, dt$. Therefore the third integral becomes,

$$\int \frac{t}{(t^2+1)^2}\, dt = \tfrac{1}{2} \int \frac{2t\, dt}{(t^2+1)^2} = \tfrac{1}{2} \int \frac{dy}{y^2} = -\tfrac{1}{2} y^{-1} + c = -\frac{1/2}{t^2+1} + c.$$

Hence our final answer is,

$$\begin{aligned} I &= \ln|t| \quad - \tfrac{1}{2} \ln|t^2+1| + \frac{1}{2(t^2+1)} + c \\ &= \ln\Big|\frac{t}{\sqrt{t^2+1}}\Big| + \frac{1}{2(t^2+1)} + c. \end{aligned}$$

4. Find the following integrals which involve top-heavy quotients. Do this by first performing a kind of long division.

(a) $\displaystyle\int_0^1 \frac{t^2+4t+5}{t+1}\, dt$, (b) $\displaystyle\int \frac{t^3+t^2-3t-5}{t^2+3t+2}\, dt$,

(c) $\displaystyle\int \frac{3t^3+t^2-t-2}{t+1}\, dt$, (d) $\displaystyle\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\, dt$.

Although it will take quite a while to solve part (d), you will know if the answer is correct because it should give you an "Oh, that's really interesting" moment.

A4.

(a) There are two ways of performing the required long division but they are, of course, entirely equivalent. One of them is what one does routinely with numerical quotients such as for $765/32$. The other works well with polynomials and here it is for the present context. The denominator is $(t+1)$ and we will convert (t^2+4t+5), a polynomial in t, into a a function of t which multiplies $(t+1)$ together with a remainder:

$$\begin{aligned} t^2+4t+5 &= t(t+1) + 3t + 5 && \text{red terms are the coefficients of } t \\ &= t(t+1) + 3(t+1) + 2 && \text{blue terms are the constants} \\ &= t(t+1) + 3(t+1) + 2. \end{aligned}$$

Hence we obtain,

$$\frac{t^2+4t+5}{t+1} = \frac{t(t+1)+3(t+1)+2}{t+1} = t + 3 + \frac{2}{t+1},$$

and so our desired integral is,

$$\begin{aligned} \int_0^1 \frac{t^2+4t+5}{t+1}\, dt &= \int_0^1 \frac{t(t+1)+3(t+1)+2}{t+1}\, dt = \int_0^1 \Big[t + 3 + \frac{2}{t+1}\Big]\, dt \\ &= \Big[\tfrac{1}{2}t^2 + 3t + 2\ln|t+1|\Big]_0^1 \\ &= \tfrac{7}{2} + 2\ln 2. \end{aligned}$$

(b) This one is a cubic divided by a quadratic, and therefore the division is a little more awkward. We seek a suitable expression for,

$$\frac{t^3+t^2-3t-5}{t^2+3t+2}.$$

Therefore,

$$t^3+t^2-3t-5=(t^2+3t+2)(t-2)+t-1$$

using the same procedure as in Q4a. Therefore,

$$\frac{t^3+t^2-3t-5}{t^2+3t+2}=t-2+\frac{t-1}{t^2+3t+2}.$$

Now we're faced with a bottom-heavy quotient where the denominator can be factorised. After a little work with partial fractions we obtain,

$$\frac{t-1}{t^2+3t+2}=\frac{t-1}{(t+1)(t+2)}=\frac{3}{t+2}-\frac{2}{t+1}.$$

Therefore the integration proceeds as follows,

$$\begin{aligned}\int\frac{t^3+t^2-3t-5}{t^2+3t+2}\,dt &= \int\Big[t-2+\frac{3}{t+2}-\frac{2}{t+1}\Big]\,dt\\ &= \tfrac{1}{2}t^2-2t+3\ln|t+2|-2\ln|t+1|+c.\end{aligned}$$

(c) We have

$$3t^3+t^2-t+2=(t+1)(3t^2-2t+1)+1,$$

and hence,

$$\frac{3t^3+t^2-t+2}{t+1}=3t^2-2t+1+\frac{1}{t+1}.$$

Hence the integral we seek is,

$$\int\frac{3t^3+t^2-t+2}{t+1}=\int\Big[3t^2-2t+1+\frac{1}{t+1}\Big]\,dt=t^3-t^2+t+\ln|t+1|+c.$$

(d) We need to start by multiplying out the numerator, and therefore the integrand becomes

$$\frac{t^4(1-t)^4}{1+t^2}=\frac{t^8-4t^7+6t^6-4t^5+t^4}{t^2+1},$$

where I have placed all the powers of t in descending order. Now we play the same game of long division as we did for part (a), followed by the appropriate simplification after division:

$$\begin{aligned}\frac{t^8-4t^7+6t^6-4t^5+t^4}{t^2+1} &= \frac{t^6(t^2+1)-4t^5(t^2+1)+5t^4(t^2+1)-4t^2(t^2+1)+4(t^2+1)-4}{t^2+1}\\ &= t^6-4t^5+5t^4-4t^2+4-\frac{4}{t^2+1}.\end{aligned}$$

Hence our desired integral is,

$$\begin{aligned}\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\,dt &= \int_0^1 \Big[t^6 - 4t^5 + 5t^4 - 4t^2 + 4 - \frac{4}{t^2+1}\Big]\,dt \\ &= \Big[\tfrac{1}{7}t^7 - \tfrac{4}{6}t^6 + t^5 - \tfrac{4}{3}t^3 + 4t - 4\tan^{-1}x\Big]_0^1 \\ &= \tfrac{22}{7} - \pi.\end{aligned}$$

That's a nice answer and my calculator reckons that it comes to 0.00126449 (6SFs). Of course many folk regard $\frac{22}{7}$ to be a good approxiamtion to π, and if one were to use that here then the integral would become zero. Given that the integrand is positive (with zero values only at $t = 0$ and $t = 1$) one must expect a positive answer, and so a zero solution is very definitely incorrect.