Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem Sheet 5 — Integration by Parts

1 Obtain the following integrals.

(a) $\int x^5 \cos ax\, dx$, (b) $\int x^5 \sin ax\, dx$, (c) $\int x^5 e^{ax}\, dx$, (d) $\int x^5 e^{ajx}\, dx$.

(e) Can you find the integral, $\int x^5 e^{-ax}\, dx$, directly from the answer to part (c)?

(f) Have you seen the real part of the answer to Q1d before? And the imaginary part?

A1. These are all integration by parts examples. For convenience and minimisation of effort I will, of course, use the 'Rees' method.

(a)
$$\int x^5 \cos bx\, dx = \Big[x^5\Big]\Big[\frac{\sin bx}{b}\Big] - \Big[5x^4\Big]\Big[-\frac{\cos bx}{b^2}\Big] + \Big[20x^3\Big]\Big[-\frac{\sin bx}{b^3}\Big]$$
$$-\Big[60x^2\Big]\Big[\frac{\cos bx}{b^4}\Big] + \Big[120x\Big]\Big[\frac{\sin bx}{b^5}\Big] - \Big[120\Big]\Big[-\frac{\cos bx}{b^6}\Big] + c$$
$$= \frac{\sin bx}{b^6}\Big[(bx)^5 - 20(bx)^3 + 120(bx)\Big] + \frac{\cos bx}{b^6}\Big[5(bx)^4 - 60(bx)^2 + 120\Big] + c.$$

(b)
$$\int x^5 \sin bx\, dx = \Big[x^5\Big]\Big[-\frac{\cos bx}{b}\Big] - \Big[5x^4\Big]\Big[-\frac{\sin bx}{b^2}\Big] + \Big[20x^3\Big]\Big[\frac{\cos bx}{b^3}\Big]$$
$$-\Big[60x^2\Big]\Big[\frac{\sin bx}{b^4}\Big] + \Big[120x\Big]\Big[-\frac{\cos bx}{b^5}\Big] - \Big[120\Big]\Big[-\frac{\sin bx}{b^6}\Big] + c$$
$$= \frac{-\cos bx}{b^6}\Big[(bx)^5 - 20(bx)^3 + 120(bx)\Big] + \frac{\sin bx}{b^6}\Big[5(bx)^4 - 60(bx)^2 + 120\Big] + c$$

(c)
$$\int x^5 e^{ax}\, dx = \Big[x^5\Big]\Big[\frac{e^{ax}}{a}\Big] - \Big[5x^4\Big]\Big[\frac{e^{ax}}{a^2}\Big] + \Big[20x^3\Big]\Big[\frac{e^{ax}}{a^3}\Big]$$
$$-\Big[60x^2\Big]\Big[\frac{e^{ax}}{a^4}\Big] + \Big[120x\Big]\Big[\frac{e^{ax}}{a^5}\Big] - \Big[120\Big]\Big[\frac{e^{ax}}{a^6}\Big] + c$$
$$= \frac{e^{ax}}{a^6}\Big[(ax)^5 - 5(ax)^4 + 20(ax)^3 - 60(ax)^2 + 120(ax) - 120\Big] + c$$

(d)
$$\int x^5 e^{ajx}\, dx = \Big[x^5\Big]\Big[\frac{e^{ajx}}{aj}\Big] - \Big[5x^4\Big]\Big[\frac{e^{ajx}}{a^2j^2}\Big] + \Big[20x^3\Big]\Big[\frac{e^{ajx}}{a^3j^3}\Big]$$
$$-\Big[60x^2\Big]\Big[\frac{e^{ajx}}{a^4j^4}\Big] + \Big[120x\Big]\Big[\frac{e^{ajx}}{a^5j^5}\Big] - \Big[120\Big]\Big[\frac{e^{ajx}}{a^6j^6}\Big] + c$$
$$= \frac{e^{ajx}}{(aj)^6}\Big[(ajx)^5 - 5(ajx)^4 + 20(ajx)^3 - 60(ajx)^2 + 120(ajx) - 120\Big] + c$$

Now to tidy up this expression and to separate it into its real and imaginary parts.

$$\begin{aligned}
\int x^5 e^{ajx} &= \frac{e^{ajx}}{(aj)^6}\Big[(ajx)^5 - 5(ajx)^4 + 20(ajx)^3 - 60(ajx)^2 + 120(ajx) - 120\Big] + c \\
&= -\frac{e^{ajx}}{a^6}\Big[j(ax)^5 - 5(ax)^4 - 20j(ax)^3 + 60(ax)^2 + 120j(ax) - 120\Big] + c \\
&= \frac{\cos ax + j\sin ax}{a^6}\Big[\Big(5(ax)^4 - 60(ax)^2 + 120\Big) - j\Big((ax)^5 - 20(ax)^3 + 120(ax)\Big)\Big] + c \\
&= \left[\frac{\sin ax}{a^6}\Big[(ax)^5 - 20(ax)^3 + 120(ax)\Big] + \frac{\cos ax}{a^6}\Big[5(ax)^4 - 60(ax)^2 + 120\Big]\right] \\
&+ \quad j\left[\frac{-\cos ax}{a^6}\Big[(ax)^5 - 20(ax)^3 + 120(ax)\Big] + \frac{\sin ax}{a^6}\Big[5(ax)^4 - 60(ax)^2 + 120\Big]\right] + c.
\end{aligned}$$

(e) Can you find the integral, $\int x^5 e^{-ax}\,dx$, directly from the answer to part (c)?

Answer: Just replace all the occurences of "a" by "$-a$":

$$\int x^5 e^{-ax}\,dx = -\frac{e^{-ax}}{a^6}\Big[(ax)^5 + 5(ax)^4 + 20(ax)^3 + 60(ax)^2 + 120(ax) + 120\Big] + c.$$

(f) Have you seen the real part of the answer to Q1d before? And the imaginary part?

Yes, they are the solutions to Q1a and Q1b, respectively, i.e. the red and blue solutions in Q1d. In fact this happens because of the following,

$$\int x^5 e^{ajx}\,dx = \int x^5\left(\cos ax + j\sin ax\right)\,dx = \int x^5\cos ax\,dx + j\int x^5 \sin ax\,dx.$$

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2. Evaluate the following integrals by integrating by parts twice.

(a) $\displaystyle\int \sin ax \sinh bx \, dx$, (b) $\displaystyle\int \sin ax \cosh bx \, dx$, (c) $\displaystyle\int_0^\infty e^{-ax} \cos \omega x \, dx$, (d) $\displaystyle\int_0^\infty e^{-ax} \sin \omega x \, dx$.

Can you think of an easy way of doing the last two integrals simultaneously which doesn't involve integration by parts? [Hint: consider the solution to Q1f.]

A2.

(a) Let $I = \displaystyle\int \sin ax \sinh bx \, dx$. We may choose either function in the integrand to integrate first. Choose the $\sinh$ function, and integrate by parts twice:

$$\begin{aligned} I &= \int \sin ax \sinh bx \, dx \\ &= \Big[\sin ax\Big]\Big[\frac{\cosh bx}{b}\Big] - \Big[a \cos ax\Big]\Big[\frac{\sinh bx}{b^2}\Big] + \int \Big[-a^2 \sin ax\Big]\Big[\frac{\sinh bx}{b^2}\Big] \, dx \\ &= \frac{1}{b^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big] - \frac{a^2}{b^2} I \\ \Longrightarrow \quad \Big(1 + \frac{a^2}{b^2}\Big) I &= \frac{1}{b^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big] \\ \Longrightarrow \quad I &= \frac{1}{a^2 + b^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big]. \end{aligned}$$

Alternatively, if one chooses to integrate the $\sin$ function first, then we obtain,

$$\begin{aligned} I &= \int \sin ax \sinh bx \, dx \\ &= \Big[\frac{-\cos ax}{a}\Big]\Big[\sinh bx\Big] - \Big[\frac{-\sin ax}{a^2}\Big]\Big[b \cosh bx\Big] + \int \Big[\frac{-\sin ax}{a^2}\Big]\Big[b^2 \cosh bx\Big] \, dx \\ &= \frac{1}{a^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big] - \frac{b^2}{a^2} I \\ \Longrightarrow \quad \Big(1 + \frac{b^2}{a^2}\Big) I &= \frac{1}{a^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big] \\ \Longrightarrow \quad I &= \frac{1}{a^2 + b^2}\Big[b \sin ax \cosh bx - a \cos ax \sinh bx\Big]. \end{aligned}$$

(b) $\displaystyle\int \sin ax \cosh bx \, dx = \frac{1}{a^2 + b^2}\Big[b \sin ax \sinh bx - a \cos ax \cosh bx\Big]$.

I've missed out the workings. Hopefully no problem there.

(c) We'll do this one in full. We have

$$\begin{aligned} I &= \int_0^\infty e^{-at}\,\cos\omega t\,dt \\ &= \left[\frac{e^{-at}}{-a}\right]\Big[\cos\omega t\Big]_0^\infty - \left[\frac{e^{-at}}{a^2}\right]\Big[-\omega\sin\omega t\Big]_0^\infty + \int_0^\infty \left[\frac{e^{-at}}{a^2}\right]\Big[-\omega^2\cos\omega t\Big]\,dt \\ &= \frac{1}{a} + 0 - \frac{\omega^2}{a^2} I \end{aligned}$$

$$\begin{aligned} \text{hence} \qquad \Big(1 + \frac{\omega^2}{a^2}\Big) I &= \frac{1}{a} \\ \Rightarrow \qquad (a^2 + \omega^2) I &= a \\ \Rightarrow \qquad I &= \frac{a}{a^2 + \omega^2}. \end{aligned}$$

Note that we could have done this by integrating the cosine first rather than the exponential. Obviously this would yield the same answer.

(d) The answer to this is,

$$I = \int_0^\infty e^{-at}\,\sin\omega t\,dt = \frac{\omega}{a^2 + \omega^2}.$$

Given our experience with Q1, we can combine the integrals in Q2c and Q2d as follows:

$$\begin{aligned} I &= \int_0^\infty e^{-at}\,\cos\omega t\,dt + j\int_0^\infty e^{-at}\,\sin\omega t\,dt \\ &= \int_0^\infty e^{-at}\Big[\cos\omega t + j\sin\omega t\Big]\,dt \\ &= \int_0^\infty e^{-(a-j\omega)t}\,dt \\ &= \frac{1}{a - j\omega} \\ &= \frac{a + j\omega}{(a - j\omega)(a + j\omega)} \\ &= \frac{a + j\omega}{a^2 + \omega^2} \\ &= \frac{a}{a^2 + \omega^2} + j\frac{\omega}{a^2 + \omega^2}. \end{aligned}$$

Therefore the real part of I yields the integral in Q2c and the imaginary part the integral in Q2d.

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3. Evaluate

(a) $\displaystyle\int \Big[\ln|x|\Big]^2 dx$, (b) $\displaystyle\int \Big[\ln|x|\Big]^3 dx$, (c) $\displaystyle\int \Big[\ln|x|\Big]^n dx$, (d) $\displaystyle\int_0^1 \frac{\ln|x|}{x^{1/2}}\, dx$ (use: $x^{1/2}\ln x \to 0$ as $x \to 0^+$),

(e) $\displaystyle\int_0^2 x^3 \ln|x|\, dx$, (f) Evaluate Q3a by first using the substitution $x = e^y$.

A3. In many of these answers I have been lazy in my presentation and have used $\ln x$ instead of $\ln|x|$. I originally had the modulus signs in, but the typesetting looked really ugly. The following is correct when $x > 0$, of course, but do replace x by $|x|$ in your mind whenever it is an argument of the ln function.

(a) This may be done by integrating by parts where the second function is 1:

$$\begin{aligned}
\int [\ln x]^2\, dx &= \int [\ln x]^2 \times 1\, dx \\
&= \Big[[\ln x]^2\Big]\Big[x\Big] - \int \Big[\frac{2\ln x}{x}\Big]\Big[x\Big]\, dx \qquad \text{(chain rule!)} \\
&= x[\ln x]^2 - 2\int \ln x\, dx \qquad \text{(on tidying up the integrand)} \\
&= x[\ln x]^2 - 2\Big[x\ln x - x\Big] \qquad \text{(using } \textstyle\int \ln x\, dx \text{ from the notes, or otherwise)} \\
&= x[\ln x]^2 - 2x\ln x + 2x \\
&= x\Big[[\ln x]^2 - 2\ln x + 2\Big].
\end{aligned}$$

(b) This follows in the same way.

$$\begin{aligned}
\int [\ln x]^3\, dx &= \int [\ln x]^3 \times 1\, dx \\
&= \Big[[\ln x]^3\Big]\Big[x\Big] - \int \Big[\frac{3[\ln x]^2}{x}\Big]\Big[x\Big]\, dx \\
&= \Big[[\ln x]^3\Big]\Big[x\Big] - 3\int [\ln x]^2\, dx
\end{aligned}$$

Now we need to evaluate the integral of the square of $\ln x$ in the same way, but this has already been done in Q3a. Eventually we will get back to

$$\int [\ln x]^3\, dx = x\Big[[\ln x]^3 - 3[\ln x]^2 + 6[\ln x] - 6\Big].$$

(c)The same approach yields

$$\int [\ln x]^n\, dx = x\Big[[\ln x]^n - n[\ln x]^{n-1} + n(n-1)[\ln x]^{n-2} \cdots + (-1)^{n-1}(n!)[\ln x] + (-1)^n(n!)\Big].$$

These integrals could also be evaluated, and possibly more easily and quickly, by first substituting $x = e^y$.

A possibly tidier way of writing down this solution is the following:

$$\int [\ln x]^n \, dx = n! \times x \left[\frac{[\ln x]^n}{n!} - \frac{[\ln x]^{n-1}}{(n-1)!} + \frac{[\ln x]^{n-2}}{(n-2)!} + \cdots + (-1)^{n-1} \frac{[\ln x]}{1!} + (-1)^n \right].$$

(d) This time we have,

$$\begin{aligned}
\int_0^1 \frac{\ln x}{x^{1/2}} \, dx &= \Big[2x^{1/2}\Big] \Big[\ln x\Big]_0^1 - \int_0^1 \Big[2x^{1/2}\Big] \Big[\frac{1}{x}\Big] \\
&= 0 - 2 \int_0^1 x^{-1/2} \, dx \qquad \text{using the given limit result and tidying the integral} \\
&= -2 \Big[2x^{1/2}\Big]_0^1 \\
&= -4.
\end{aligned}$$

We expect a negative result because $\ln x$ is negative within the range of integration.

(e) We have,

$$\begin{aligned}
\int_0^2 x^3 \ln x \, dx &= \Big[\frac{x^4}{4}\Big] \Big[\ln x\Big]_0^2 - \int_0^2 \Big[\frac{x^4}{4}\Big] \Big[\frac{1}{x}\Big] \, dx \\
&= 4 \ln 2 - \int_0^2 \frac{x^3}{4} \, dx \qquad (x^4 \ln x = 0 \text{ when } x = 0) \\
&= 4 \ln 2 - \Big[\frac{x^4}{16}\Big]_0^2 \\
&= 4 \ln 2 - 1.
\end{aligned}$$

(f) The substitution is $x = e^y$ and therefore $dx = e^y \, dy$. The integral becomes,

$$\begin{aligned}
\int [\ln x]^2 \, dx &= \int y^2 e^y \, dy \\
&= (y^2 - 2y + 2) e^y + c \qquad \text{(integration by parts)} \\
&= x \Big[[\ln x]^2 - 2 \ln x + 2 \Big] + c \qquad \text{(returning to } x\text{)}
\end{aligned}$$

Although Q3a uses integration parts as well, it has to execute each of them one-by-one and therefore it is slow relative to this method which starts with a substitution.

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4. The mandatory silly integral: $\int \sin(x)\ln\Big[\tan\Big(\frac{x}{2}+\frac{\pi}{4}\Big)\Big]\,dx.$

A4. Yes, this one is silly from the point of view that there is no way one is going to attempt to integrate the log function. However, one can differentiate it. Using the chain rule we obtain,

$$\begin{aligned}\frac{d}{dx}\ln\left(\tan(\frac{x}{2}+\frac{\pi}{4})\right) &= \frac{1}{\tan(x/2+\pi/4)}\times\sec^2(x/2+\pi/4)\times\tfrac{1}{2}\\ &= \frac{\cos(x/2+\pi/4)}{2\sin(x/2+\pi/4)\cos^2(x/2+\pi/4)}\\ &= \frac{1}{2\sin(x/2+\pi/4)\cos(x/2+\pi/4)} = \frac{1}{\sin(x+\pi/2)} = \frac{1}{\cos x}.\end{aligned}$$

We are now in a position to perform the integration:

$$\begin{aligned}\int\sin(x)\ln\left(\tan(\frac{x}{2}+\frac{\pi}{4})\right)dx &= \Big[-\cos x\Big]\Big[\ln\left(\tan(\frac{x}{2}+\frac{\pi}{4})\right)\Big]-\int\Big[-\cos x\Big]\Big[\frac{1}{\cos x}\Big]\,dx\\ &= -(\cos x)\ln\left(\tan(\frac{x}{2}+\frac{\pi}{4})\right)+x+c.\end{aligned}$$

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5. Use integration by parts once to obtain a formula for $I_n = \int_0^\infty x^n e^{-ax}\,dx$ in terms of I_{n-1} — such a formula is called a **Reduction Formula** or a **Recurrence Relation**. Find I_0 directly, and use the reduction formula to evaluate I_6. Check your answer by evaluating I_6 using integration by parts.

A5. Doing what is asked we get

$$\begin{aligned}I_n &= \int_0^\pi x^n e^{-ax}\,dx\\ &= \Big[x^n\Big]\Big[-e^{-ax}/a\Big]_0^\infty - \int_0^\infty\Big[nx^{n-1}\Big]\Big[-e^{-ax}/a\Big]\,dx\\ &= 0+(n/a)\int_0^\infty x^{n-1}e^{-ax}\,dx\\ &= (n/a)I_{n-1}.\end{aligned}$$

Therefore we have an expression for I_n in terms of I_{n-1}; this is what is called a **recurrence relation** or even a **reduction formula**. Now for $n=0$ we have,

$$I_0 = \int_0^\infty e^{-ax}\,dx = 1/a.$$

Therefore $I_0 = 1/a$, $I_1 = 1/a^2$, $I_2 = (2/a)I_1 = 2/a^3$, $I_3 = 3I_2 = (6/a^4) = 3!/a^4$, and so on. Hence

$$I_n = \frac{n!}{a^{n+1}}.$$

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6. Something weird. Evaluate the following indefinite integral using one integration by parts where you'll choose to integrate f' first:

$$I = \int \frac{1}{f}\, f'\, dx.$$

Can you explain why the answer is incorrect? What happens if you choose to integrate between $x = a$ and $x = b$?

A6. Let's do it and see what happens....

$$\begin{aligned} I &= \int \frac{1}{f}\, f'\, dx \qquad \text{as given} \\ &= \Big[\frac{1}{f}\Big]\Big[f\Big] - \int \Big[-\frac{f'}{f^2}\Big]\Big[f\Big]\, dx \qquad \text{(I by P done)} \\ &= 1 + \int \frac{1}{f}\, f'\, dx \qquad \text{(after simplification)} \\ &= 1 + I. \end{aligned}$$

Therefore $I = 1 + I$ which means that $0 = 1$. So what has gone stupidly wrong here?

We've omitted the arbitrary constants. That might sound glib, but all indefinite integrals will have an infinite number of solutions and they differ solely by the value of the arbitrary constant.

OK, so what happens with the definite integral case? We get,

$$\begin{aligned} I &= \int_a^b \frac{1}{f}\, f'\, dx \qquad \text{as given} \\ &= \Big[\frac{1}{f}\Big]\Big[f\Big]_a^b - \int_a^b \Big[-\frac{f'}{f^2}\Big]\Big[f\Big]\, dx \qquad \text{(I by P done)} \\ &= \Big[1\Big]_a^b + \int_a^b \frac{1}{f}\, f'\, dx \qquad \text{(after simplification)} \\ &= 0 + I. \end{aligned}$$

So we don't get much except that $I = I$, so at least the procedure is self-consistent.

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7. Also weird. Define $I(a)$ according to $I(a) = \displaystyle\int_0^\infty e^{-ax}\, dx$, and evaluate this integral. Clearly the value taken by $I(a)$ depends on a, and therefore we can differentiate it with respect to a. Do this and find $I'(a)$ both as an integral and as a function of a. Continue to differentiate in this way with the aim of eventually finding $\displaystyle\int_0^\infty x^4 e^{-ax}\, dx$.

Using this idea and the solution to Q2c, find $\displaystyle\int_0^\infty x e^{-ax} \cos\omega x\, dx$, the integral of the product of three functions.

A7. We find that,

$$I(a)=\int_0^\infty e^{-ax}\,dx=\Big[\frac{e^{-ax}}{-a}\Big]_0^\infty=\frac{1}{a}.$$

We note that the value that I takes depends on the chosen value of a, and therefore I is a function of a. So now we shall take the a-derivative of the above equation:

$$I'(a)=\int_0^\infty -xe^{-ax}\,dx=-\frac{1}{a^2}. \tag{1}$$

Another a-derivative gives,

$$I''(a)=\int_0^\infty x^2e^{-ax}\,dx=\frac{2}{a^3}.$$

Two more:

$$I'''(a)=\int_0^\infty -x^3e^{-ax}\,dx=-\frac{6}{a^4},$$

$$I''''(a)=\int_0^\infty x^4e^{-ax}\,dx=\frac{24}{a^5}.$$

We have, of course, assumed that the processes of integration with respect to x and differentiation with respect to a may be done in either order; this is generally true. Given the way that the coefficients have accumulated on the far right hand side of each integral we can say that,

$$\int_0^\infty x^4e^{-ax}\,dx=\frac{4!}{a^5}.$$

The integral, $\int_0^\infty xe^{-ax}\cos\omega x\,dx$, may now be evaluated by taking the a-derivative of the result of Q2c. That result is,

$$J(a,\omega)=\int_0^\infty e^{-ax}\cos\omega x\,dx=\frac{a}{a^2+\omega^2}.$$

Therefore,

$$\frac{\partial J}{\partial a}=\int_0^\infty -xe^{-ax}\cos\omega x\,dx=\frac{\partial}{\partial a}\Big[\frac{a}{a^2+\omega^2}\Big]=\frac{\omega^2-a^2}{(a^2+\omega^2)^2}.$$

I have used partial derivative notation here — I shall introduce you formally to it next semester — because J is a function of both a and ω. Hence,

$$\int_0^\infty xe^{-ax}\cos\omega x\,dx=\frac{a^2-\omega^2}{(a^2+\omega^2)^2}.$$

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8. Evaluate the mean and RMS of the following functions

(a) $f(t)=t^2 \quad (0\le t\le 1)$,

(b) $f(t)=\sin t \qquad (0\le t\le 2\pi)$,

(c) $f(t)=|\sin t| \qquad (0\le t\le 2\pi)$,

(d) $f(t)=e^{-t} \qquad (0\le t\le 1)$.

A8.

(a) The mean and RMS are

$$\text{mean}=\int_0^1 t^2\,dt=\tfrac{1}{3}.$$

$$\text{RMS} = \sqrt{\int_0^1 t^4\, dt} = \sqrt{\tfrac{1}{5}}.$$

(b) The mean and RMS are

$$\text{mean} = \frac{1}{2\pi}\int_0^{2\pi} \sin t\, dt = 0.$$

$$\text{RMS} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\, dt} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt} = \sqrt{\frac{1}{2\pi}\times\pi} = \frac{1}{\sqrt{2}}.$$

(c) For the next function, which is a rectified sine wave, we can consider the standard sine wave between 0 and π as the function repeats exactly outside of this range. Hence

$$\text{mean} = \frac{1}{\pi}\int_0^{\pi} \sin t\, dt = \frac{1}{\pi}\Big[-\cos t\Big]_0^{\pi} = \frac{2}{\pi}.$$

$$\text{RMS} = \sqrt{\frac{1}{\pi}\int_0^{\pi} \sin^2 t\, dt} = \sqrt{\frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2t}{2}\, dt} = \sqrt{\frac{1}{\pi}\times\pi/2} = \frac{1}{\sqrt{2}}.$$

Thus the RMS values of the sine wave and of the rectified sine wave are equal.

(d) The mean is given by

$$\text{mean} = \int_0^1 e^{-t}\, dt = 1 - e^{-1} \simeq 0.3679.$$

The RMS is

$$\text{RMS} = \sqrt{\int_0^1 e^{-2t}\, dt} = \sqrt{\tfrac{1}{2}(1-e^{-2})} \simeq 0.6575.$$

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9. This is a set of miscellaneous questions and you may wish to wait until the revision period to tackle them.

I won't give too many hints about how to do them! Sorry. Some are substitutions but (e) should be done both as an integration by parts and by using a suitable multiple angle formula. Some will require the use of more than one method. Questions (g) and (h) are trick questions — it all depends on how quickly you can see the trick. As a guide (or perhaps a challenge!), when I saw (h) for the very first time (on a youtube video) I managed 5 seconds, and so it was immediately marked up as something that I just had to try out on you!

(a) $\displaystyle\int_0^{\pi^2} \sin\sqrt{x}\, dx$, (b) $\displaystyle\int_0^{\infty} e^{-\sqrt{x}}\, dx$,

(c) $\displaystyle\int_0^{\infty} e^{-x^{1/4}}\, dx$, (d) $\displaystyle\int_1^{e} \sin(\ln x)\, dx$,

(e) $\displaystyle\int_0^{\pi/2} \sin x\cos 5x\, dx$, (f) $\displaystyle\int \frac{x+3}{\sqrt{x^2+6x+10}}\, dx$,

(g) $\displaystyle\int_{-\infty}^{\infty} e^{-x^4}\sin 3x\,dx$, (h) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx.$

(i) $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$, (j) $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$, (k) $\displaystyle\int_0^{\infty} \frac{1}{1+e^x}\,dx.$

A9. In the following I will try to give some reasoning about I would approach these cases.

(a) The principle here is to remove the square root within the sine and hopefully the mess that accrues isn't too bad. So, we let $x = y^2$. The bookkeeping is:

$$x = y^2 \implies dx = 2y\,dy, \qquad x = 0 \implies y = 0, \qquad x = \pi^2 \implies y = \pi.$$

Hence,

$$\begin{aligned} I &= \int_0^{\pi^2} \sin\sqrt{x}\,dx \\ &= \int_0^{\pi} (\sin y)\,2y\,dy \qquad \text{so we need to integrate by parts} \\ &= 2\Big[[-\cos y][y] - [-\sin y][1]\Big]_0^{\pi} \\ &= 2\pi. \end{aligned}$$

(b)This one looks a bit like the previous one so we'll use the same trick. The bookkeeping is,

$$x = y^2 \implies dx = 2y\,dy, \qquad x = 0 \implies y = 0, \qquad x \to \infty \implies y \to \infty.$$

Hence,

$$\begin{aligned} I = &\int_0^{\infty} e^{\sqrt{x}}\,dx \\ = &\int_0^{\infty} e^{y}\,2y\,dy \qquad \text{again integration by parts} \\ = &\ 2\Big[[-e^{-y}][y] - [e^{-y}][1]\Big]_0^{\infty} \\ = &\ 2. \end{aligned}$$

(c) This too displays a similarity to the previous question, so we'll follow the same sort of approach (noting that this won't work should the integrand have been e^{-y^2} or e^{-y^4}). So we'll let $y = x^{1/4}$. The bookkeeping is,

$$x = y^4 \implies dx = 4y^3\,dy \qquad x = 0 \implies y = 0, \qquad x \to \infty \implies y \to \infty.$$

Hence,

$$
\begin{aligned}
I = & \int_0^\infty e^{-x^{1/4}}\,dx \\
= & \int_0^\infty e^{-y}\,4y^3\,dy \qquad \text{(yes, integration by parts again)} \\
= & \; 4\Big[[-e^{-y}][y^3] - [e^{-y}][3y^2] + [-e^{-y}][6y] - [e^{-y}][6]\Big]_0^\infty \\
= & \; 24.
\end{aligned}
$$

Given how these last two have worked out, I would guess that $\displaystyle\int_0^\infty e^{-x^{1/n}}\,dx$, where n is a positive integer, has the value, $n!$. Do please check this!

(d) We'll remove the log from within the sine by setting $x = e^y$, and then we'll see if it i possible to proceed. Hence,

$$x = e^y \implies dx = e^y\,dy \qquad x = 1 \implies y = 0, \qquad x = e \implies y = 1,$$

and so,

$$
\begin{aligned}
\int_1^e \sin(\ln x)\,dx \; &= \int_0^1 \sin y\, e^y\,dy \qquad \text{(will try the complex number route for a change)} \\
&= \text{Imag} \int_0^1 e^y\, e^{jy}\,dy = \text{Imag} \int_0^1 e^{(1+j)y}\,dy \\
&= \text{Imag}\Big[\frac{1}{1+j} e^{(1+j)y}\Big]_0^1 \\
&= \text{Imag}\Big[\tfrac{1}{2}(1-j)\Big(e^{(1+j)} - 1\Big)\Big] \qquad \text{(using complex conjugate)} \\
&= \text{Imag}\Big[\tfrac{1}{2}(1-j)\Big(e(\cos 1 + j\sin 1) - 1\Big)\Big] \\
&= \tfrac{1}{2}\Big[e(\sin 1 - \cos 1) - 1\Big] \qquad \text{(having taken the imaginary part)}
\end{aligned}
$$

This approach of letting $\sin y = \text{Imag}\,[\,e^{jy}\,]$ can be quicker than when using integration by parts, despite the number of lines used above.

(e) As requested, we'll do it in two different ways. The first will involve the use of multiple-angle formulae.

We'll use

$$\sin ax \cos bx = \tfrac{1}{2}\Big[\sin(a+b)x + \sin(a-b)x\Big],$$

and so our case reads:

$$\sin x \cos 5x = \tfrac{1}{2}\Big[\sin 6x - \sin 4x\Big].$$

Therefore

$$I = \int_0^{\pi/2} \sin x \cos 5x = \tfrac{1}{2}\int_0^{\pi/2} \Big[\sin 6x - \sin 4x\Big]\,dx \; = \tfrac{1}{2}\Big[-\tfrac{1}{6}\cos 6x + \tfrac{1}{4}\cos 4x\Big]_0^{\pi/2} = \tfrac{1}{6}.$$

Alternatively, we may attempt integration by parts.

$$\begin{aligned} I &= \int_0^{\pi/2} \sin x \cos 5x \\ &= \Big[-\cos x\Big]\Big[\cos 5x\Big]_0^{\pi/2} - \Big[-\sin x\Big]\Big[-5\sin 5x\Big]_0^{\pi/2} + \int_0^{\pi/2}\Big[-\sin x\Big]\Big[-25\cos 5x\Big]\,dx \\ &= (0-(-1)) - (5-0) + 25I \end{aligned}$$

$$\Longrightarrow\ -24I = -4 \qquad \Longrightarrow\ I = \tfrac{1}{6}.$$

Just to note that Integration by Parts may be used whenever one has a product of sines, or of cosines, or of one of each. The method won't work when the coefficients of x in the sinusoids are equal; Examples include $\sin 3x \cos 3x$. It is worth attempting the following to see the manner in which Integration by Parts fails:

$$\int_0^{\pi/2} \sin 3x \cos 3x\, dx.$$

(f) If this didn't have the square root in the denominator, then we could have used *completion of the square* and then followed the usual tricks for when one has a linear factor divided by an irreducible quadratic. Perhaps we might as well start the same way and see what happens.

Given that $x^2 + 6x + 10 = (x+3)^2 + 1$, we can let $x + 3 = \tan\theta$. Hence $dx = \sec^2\theta\, d\theta$. The integral becomes,

$$\begin{aligned} I &= \int \frac{x+3}{\sqrt{x^2+6x+10}}\,dx \\ &= \int \frac{\tan\theta}{\sqrt{\tan^2\theta + 1}}\sec^2\theta\, d\theta, \\ &= \int \tan\theta\ \sec\theta\, d\theta \qquad \text{(I'll assume that I don't know that this integral is } \sec\theta) \\ &= \int \frac{\sin\theta}{\cos^2\theta}\,d\theta \\ &= \int -\frac{1}{z^2}\,dz \ = \ \frac{1}{z} \qquad \text{(using } z = \cos\theta \text{ substitution)} \\ &= \frac{1}{\cos\theta} \ = \ \sec\theta \\ &= \sqrt{\sec^2\theta} \\ &= \sqrt{1+\tan^2\theta} \\ &= \sqrt{x^2+6x+10}. \end{aligned}$$

Now that looks fishy because we have reproduced the original denominator. This suggests that there may have been an easier way of doing this. Actually, that easier method is related to the so-called f'/f method.

The present integrand is actually of the form, $f'/\sqrt{f}$, and we can use this to determine a much better initial substitution: let $v = x^2 + 6x + 10$. Given that $dv = (2x^2 + 6x)\,dx$, or $\frac{1}{2}dv = (x + 3)\,dx$, we have,

$$I = \int \frac{\frac{1}{2}}{\sqrt{v}}\,dv = \sqrt{v} + c = \sqrt{x^2 + 6x + 10} + c.$$

In general, we may wish to evaluate $\displaystyle\int \frac{f'}{\sqrt{f}}\,dx$ where $f = f(x)$ is an unspecified function. Let $y = \sqrt{f}$, and hence,

$$dy = \frac{f'}{2\sqrt{f}}dx.$$

Therefore,

$$\int \frac{f'}{\sqrt{f}}\,dx = \int 2\,dy = 2y + c = 2\sqrt{f} + c.$$

The big question: can this result be generalised further?

(g) The integral is zero because the integrand is odd and the range of integration is symmetric.

(h) The way that I worked out the solution in my head involved the following two thoughts: (i) the denominator is even about $x = \pi/4$, and (ii) therefore the value of the integral would be exactly the same if I were to replace the numerator by $\sqrt{\cos x}$. Given that we now have two integrals with exactly the same value we might as well add them together:

$$\begin{aligned} & \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \\ = & \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \\ = & \int_0^{\pi/2} 1\,dx \\ = & \tfrac{1}{2}\pi. \end{aligned}$$

Hence the required value is half of that, $\frac{1}{4}\pi$.

(i) We'll get rid of the square root in the denominator, so we'll let $x = y^2$. Hence the bookkeeping is,

$$x = y^2 \implies dx = 2y\,dy, \qquad x = 0 \implies y = 0, \qquad x = 1 \implies y = 1.$$

Hence,

$$\begin{aligned}
I &= \int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}\,dx \\
&= \int_0^1 \frac{\sqrt{1-y^2}}{y}\,2y\,dy \\
&= 2\int_0^1 \sqrt{1-y^2}\,dy \qquad \text{(now use } y=\sin\theta\text{)} \\
&= 2\int_0^{\pi/2} \cos\theta\,\cos\theta\,d\theta \qquad \text{(also note the limits have been changed)} \\
&= \int_0^{\pi/2} \Big(\cos 2\theta + 1\Big)\,d\theta \qquad \text{(double angle formula)} \\
&= \Big[\frac{\sin 2\theta}{2} + \theta\Big]_0^{\pi/2} \\
&= \tfrac{1}{2}\pi.
\end{aligned}$$

(j) An unusual denominator, but this one makes me think of how we use the complex conjugate when dividing a complex number by another. So we will start by multiplying both the numerator and the denominator by a suitable function.

$$\begin{aligned}
I &= \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx \\
&= \int \frac{(\sqrt{x+1}-\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}\,dx \\
&= \int \frac{(\sqrt{x+1}-\sqrt{x})}{(x+1)-(x)}\,dx \\
&= \int \Big[\sqrt{x+1}-\sqrt{x}\Big]\,dx \\
&= \tfrac{2}{3}\Big[(x+1)^{3/2} - x^{3/2}\Big] + c.
\end{aligned}$$

(k) We'll let let $y = 1 + e^x$ for there doesn't seem to be anything else that suggests itself. The bookkeeping:

$$y = 1+e^x \implies dy = e^x\,dx \implies dx = dy/(y-1), \qquad x = 0 \implies y = 2, \qquad x\to\infty \implies y\to\infty.$$

The integral becomes,

$$\begin{aligned}\int_0^\infty \frac{1}{1+e^x}\,dx &= \int_2^\infty \frac{1}{(y-1)y}\,dy \\ &= \int_2^\infty \Big[\frac{1}{y-1} - \frac{1}{y}\Big]\,dy \\ &= \Big[\ln\Big(\frac{y-1}{y}\Big)\Big]_2^\infty \\ &= -\ln\tfrac{1}{2} \\ &= \ln 2.\end{aligned}$$

The indefinite integral corresponding to the integrand is

$$\ln\frac{e^x}{e^x+1}, \qquad \text{or} \qquad \ln\frac{1}{1+e^{-x}}.$$