Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Sheet 10 — Laplace Transforms

It is normal in questions on Laplace Transforms to have ready access to the LTs of functions like sinusoids, exponentials and powers. Here, though, I will need you to derive these results.

1. Find the Laplace Transforms of the following functions using the definition of the Laplace Transform (rather than by looking up the result in a table):

(a) e^{3t} (b) e^{-3t} (c) $\cos\omega t$ (d) te^{-3t} (e) t^3 (f) $t\cos\omega t$ (g) $f'''(t)$

(h) The unit pulse: $f(t) = 1$ for $t < 1$, $f(t) = 0$ otherwise (i) $\cosh\omega t$ (j) t^2e^{-t}

(k) $t^{-1/2}$ [Hint: set $x = (st)^{1/2}$ to transform the integral and use the result $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$.]

A1. Find the Laplace Transforms of the following functions using the definition of the Laplace Transform (rather than by looking up the result in a table):

(a) $\mathcal{L}[e^{3t}] = \int_0^\infty e^{3t}e^{-st}dt = \int_0^\infty e^{-(s-3)t}dt = \frac{1}{3-s}\Big[e^{-(s-3)t}\Big]_0^\infty = \frac{1}{s-3}.$

Note that this result relies on the fact that $s > 3$; when $s \leq 3$ the result is infinite and we say that the Laplace Transform does not exist in this case.

(b) $\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$ by analogy with (a). Here we require $s > -3$ for the transform to exist.

(c) $\mathcal{L}[\cos\omega t] = \int_0^\infty (\cos\omega t)e^{-st}dt = \frac{s}{s^2+\omega^2}$ using integration by parts. Here we require $s > 0$ for the Transform to exist.

(d) $\mathcal{L}[te^{-3t}] = \int_0^\infty te^{-(s+3)t}dt = \frac{1}{(s+3)^2}$ using integration by parts. Again $s > -3$ for the Transform to exist.

(e) $\mathcal{L}[t^3] = \frac{6}{s^4}.$ Four integrations by parts.

(f) $\mathcal{L}[t\ \cos\omega t]$ will, when written out as the full integral,

$$\int_0^\infty t\ \cos(\omega t)\ e^{-st},$$

be found to have an integrand which is the product of three functions. This can integrated but it is a very long process and a real hassle which involves finding other integrals. However, it may be integrated much more quickly by changing $\cos\omega t$ to $\exp j\omega t$. Then we combine the two exponentials, undertake the integration by parts and evaluating the result between the limits, and then taking the real part of the final answer. For the sake of brevity I'll miss out the integration by parts here but this process may be summarised as follows.

$$\begin{aligned}
\int_0^\infty t\,\cos(\omega t)\,e^{-st} &= \text{Re}\,\textstyle\int_0^\infty t e^{j\omega t} e^{-st}\,dt \\
&= \text{Re}\int_0^\infty t\,e^{-(s-j\omega)t}\,dt \\
&= \text{Re}\,\frac{1}{(s-j\omega)^2} \qquad\qquad \text{using integration by parts} \\
&= \text{Re}\,\frac{(s+j\omega)^2}{(s^2+\omega^2)^2} \\
&= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.
\end{aligned}$$

An alternative way to do this is by first noting that

$$\int_0^\infty \sin(\omega t)\,e^{-st}\,dt = \frac{\omega}{s^2+\omega^2}.$$

Now differentiate both sides with respect to ω. The left and side gives,

$$\frac{\partial}{\partial\omega}\left[\int_0^\infty \sin(\omega t)\,e^{-st}\,dt\right] = \int_0^\infty t\,\cos(\omega t)\,e^{-st}\,dt,$$

while the right hand side yields,

$$\frac{\partial}{\partial\omega}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

(g) $\mathcal{L}[f'''(t)] = -f''(0) - sf'(0) - s^2 f(0) + s^3 F(s)$, where $F(s) = \mathcal{L}[f(t)]$. Three integrations by parts for this one.

(h) $\mathcal{L}[f(t)] = (1-e^{-s})/s$. Should be an easy integral!

(i) $\mathcal{L}[\cosh\omega t] = \frac{1}{2}\left(\frac{1}{s+\omega} + \frac{1}{s-\omega}\right)$. This follows quickly from $\cosh\omega t = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$.

(j) $\mathcal{L}[t^2 e^{-t}]$ should come to $2/(s+1)^3$. Three integrations by parts.

(k) Let $I = \mathcal{L}[t^{-1/2}] = \displaystyle\int_0^\infty t^{-1/2} e^{-st}dt.$ Now let $x = (st)^{1/2}$.
Hence $dx = (s^{1/2}/2t^{1/2})dt$, or, $(2/s^{1/2})dx = t^{-1/2}dt$, and therefore the integral becomes,

$$I = \int_0^\infty \frac{2}{s^{1/2}} e^{-x^2} dx = \frac{2}{s^{1/2}}\int_0^\infty e^{-x^2} dx = \frac{2}{s^{1/2}}\frac{\pi^{1/2}}{2} = \frac{\pi^{1/2}}{s^{1/2}}$$

on using the given result that

$$\int_0^\infty e^{-x^2} dx = \frac{\pi^{1/2}}{2}.$$

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2. Use the Laplace Transform to solve the following equations:

(a) $\dfrac{dy}{dt} + 4y = 6, \qquad y(0) = 2.$

(b) $\dfrac{d^2y}{dt^2} + 16y = 0, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = 1.$

(c) $\dfrac{d^2y}{dt^2} + 4y = 29e^{-5t}, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = -3.$

(d) $y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = 3, \quad y''(0) = -5.$

[You may also practice on any of the linear constant coefficient equations from the ODEs section of the unit, but note that there may be some awkwardnesses due to the fact that (i) the questions weren't designed for nice LT solutions, (ii) many don't have initial conditions specified, (iii) some of the results derived in the 3rd and 4th Laplace Transform lectures may be of considerable use.]

A2. Use the Laplace Transform to solve the following equations:

(a) $\dfrac{dy}{dt} + 4y = 6, \quad y(0) = 2.$

$\mathcal{L}[y'] = sY - y(0) = sY - 2.$ Hence the ODE reduces to $(sY - 2) + 4Y = 6/s$, from which Y may be shown to be

$$Y = \frac{2(s+3)}{s(s+4)} \equiv \frac{A}{s} + \frac{B}{s+4}.$$

Multiplying by $s(s+4)$ we obtain $2s + 6 = A(s+4) + Bs$. When $s = 0$ we see that $A = 3/2$, and when $s = -4$ we get $B = 1/2$. Hence

$$Y = \tfrac{1}{2}\left[\frac{3}{s} + \frac{1}{s+4}\right] \qquad \Rightarrow \qquad y = \tfrac{1}{2}[3 + e^{-4t}].$$

(b) $\dfrac{d^2y}{dt^2} + 16y = 0, \quad y(0) = 0, \quad \dfrac{dy}{dt} = 1.$

$\mathcal{L}[y''] = s^2Y - y'(0) - sy(0) = s^2Y - 1.$ The equation reduces to

$$(s^2Y - 1) + 16Y = 0 \qquad \Rightarrow \qquad Y = \frac{1}{s^2+16} = \tfrac{1}{4}\left[\frac{4}{s^2+4^2}\right] \qquad \Rightarrow \qquad y = \tfrac{1}{4}\sin 4t$$

using tables.

(c) $\dfrac{d^2y}{dt^2} + 4y = 29e^{-5t}, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = -3.$

The equation becomes,

$$s^2Y + 3 + 4Y = \frac{29}{s+5} \qquad \Rightarrow \qquad Y = \frac{29}{(s+5)(s^2+4)} - \frac{3}{s^2+4}.$$

After some partial fractions work, this simplifies to,

$$Y = \frac{1}{s+5} + \frac{2}{s^2+4} - \frac{s}{s^2+4}.$$

After using standard results for the LTs of exponentials and sinusoids, we obtain,

$$y = e^{-5t} + \sin 2t - \cos 2t.$$

(d) $y''' + y'' + 4y' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = 3, \quad y''(0) = -5.$

Using the standard results for the LTs of the various derivatives, we get,

$$\left[s^3 Y - y''(0) - sy'(0) - s^2 y(0)\right] + \left[s^2 Y - y'(0) - sy(0)\right] + 4\left[sY - y(0)\right] + 4Y = 0.$$

Using the initial conditions and simplifying, we get,

$$(s^3 + s^2 + 4s + 4)Y = 3s - 2.$$

The cubic multiplying Y may be factorised and therefore we have,

$$Y = \frac{3s-2}{(s^2+4)(s+1)}.$$

Standard partial fractions of the form,

$$\frac{3s-2}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1},$$

yields $A = 1$, $B = 2$ and $C = -1$. Therefore,

$$\begin{aligned} Y &= \frac{s+2}{s^2+4} - \frac{1}{s+1}, \\ &= \frac{s}{s^2+4} + \frac{2}{s^2+4} - \frac{1}{s+1}. \end{aligned}$$

All three of these terms have standard inverse LTs. Therefore we have,

$$y = \cos 2t + \sin 2t - e^{-t}.$$

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3. Find the Laplace Transform of $z(t) = \int_0^t y(\tau)\, d\tau$. [Hint: recall that $z'(t) = y(t)$ here.]

A3. We'll integrate by parts once only and differentiate the z-term:

$$\begin{aligned} \mathcal{L}\Big[\int_0^t y(\tau)\, d\tau\Big]\, dt = \mathcal{L}[z(t)] &= \int_0^\infty z(t) e^{-st}\, dt \\ &= [z] \left[\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty [z'] \left[\frac{e^{-st}}{-s}\right] dt \\ &= 0 + \frac{1}{s}\int_0^\infty y e^{-st}\, dt = \frac{Y}{s} \end{aligned}$$

In the above, note that $z(0)=0$, given its definition as an integral, i.e. that

$$z(0)=\int_0^0 y(\tau)\,d\tau.$$

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4. Find the solution of the ODE, $y''+2y'+y=2e^{-t}$, subject to $y(0)=y'(0)=0$. [Hint: you may need to consult the solution to Q1j.]

A4. The Laplace Transform of the ODE yields,

$$(s^2+2s+1)Y=\frac{2}{s+1} \qquad \Longrightarrow \qquad Y=\frac{2}{(s+1)^3}.$$

Question 1j has the solution, $\mathcal{L}[t^2e^{-1}]=2/(s+1)^3$, therefore the present solution is

$$y=t^2e^{-t}. \tag{1}$$

In the language of ODE theory we could say that the left hand side of the ODE is equivalent to $\lambda=-1,-1$ while the right hand side is equivalent to $\lambda=-1$. Hence the general CF/PI solution is,

$$y=\underbrace{(A+Bt)e^{-t}}_{\lambda=-1,-1}+\underbrace{Ct^2e^{-t}}_{\lambda=-1}, \tag{2}$$

where $C=1$ if found after substitution of (2) into the original equation. Application of the initial conditions to (2) then yields $A=B=0$, thereby reproducing the solution in (1).

A rhetorical question: which of these two solution methods do you prefer?

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5. Factorise the denominator of the following fractions into complex factors, and use partial fractions to find their Inverse Laplace Transforms: [**Note:** that I won't expect such complex factorisation in the exam.]

(a) $\dfrac{1}{s^2+b^2}$ (b) $\dfrac{s}{s^2+b^2}$ (c) $\dfrac{1}{s^2+2cs+c^2+d^2}$ (d) $\dfrac{s+c}{s^2+2cs+c^2+d^2}$.

These results may be used to solve the following equations:

(e) $y''+4y'+5y=0,\quad y(0)=0,\quad y'(0)=1.$

(f) $y''+2y'+2y=e^{-t},\quad y(0)=0,\quad y'(0)=0.$

A5. (a) $$\frac{1}{s^2+b^2}=\frac{1}{(s+bj)(s-bj)}=\frac{1}{2bj}\Big[\frac{1}{s-bj}-\frac{1}{s+bj}\Big].$$

Hence

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2+b^2}\right] &= \frac{1}{2bj}\mathcal{L}^{-1}\left[\frac{1}{s-bj}-\frac{1}{s+bj}\right]\\ &= \frac{1}{2bj}[e^{bjt}-e^{-bjt}]\\ &= \frac{1}{2bj}\Big[(\cos bt+j\sin bt)-(\cos bt-j\sin bt)\Big]=\frac{\sin bt}{b}.\end{aligned}$$

(b) $\dfrac{s}{s^2+b^2} = \dfrac{1}{2}\Big[\dfrac{1}{s-bj} + \dfrac{1}{s+bj}\Big]$.
Hence

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \tfrac{1}{2}\Big[e^{bjt}+e^{-bjt}\Big] = \cos bt.$$

(c) $\dfrac{1}{s^2+2cs+c^2+d^2} = \dfrac{1}{2dj}\Big[\dfrac{1}{s+c-dj} - \dfrac{1}{s+c+dj}\Big]$.
Note that this exercise is very similar to part (a). The solution is $(1/d)e^{-ct}\sin dt$.

(d) $\dfrac{s+c}{s^2+2cs+c^2+d^2} = \dfrac{1}{2}\Big[\dfrac{1}{s+c-dj} + \dfrac{1}{s+c+dj}\Big]$. The solution is $e^{-ct}\cos dt$.

Now we may solve the original equations:

(e) $y''+4y'+5y=0$, $y(0)=0$, $y'(0)=1$.

Here we have

$$\mathcal{L}[y''] = s^2Y - sy(0) - y'(0) = s^2Y - 1, \qquad \mathcal{L}[y'] = sY - y(0) = sY.$$

The equation transforms to $(s^2+4s+5)Y = 1$ and hence $Y = 1/(s^2+4s+5)$. Now we can use the result of part (c) with $c=2$ and $d=1$:

$$y = e^{-2t}\sin t.$$

(f) $y''+2y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=0$.

Here we find that $\mathcal{L}[y'] = sY$ and $\mathcal{L}[y''] = s^2Y$. The equation transforms to $(s^2+2s+2)Y = 1/(s+1)$, from which we obtain

$$Y = \frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} - \frac{s+1}{s^2+2s+2} \qquad \Rightarrow \qquad y = (1-\cos t)e^{-t}$$

after using partial fractions and part (d) with $c=d=1$.

Final remark: I would not recommend that such inverse transforms are undertaken using complex factorisation. However, it is nice to know that it is possible. I will not be asking for such complex factorisation in the exam.

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6. Write down the values of the following integrals.

$$\int_{-\infty}^{\infty}\delta(t)e^{2t}\,dt, \qquad \int_{-\infty}^{\infty}\delta(t-1)e^{-t^2}\,dt, \qquad \int_{-\infty}^{\infty}\delta(t-2)\sin\pi t\,dt, \qquad \int_{0}^{\infty}\delta(t+2)t^3\,dt.$$

A6. In all cases we apply the result that

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)\,dt = g(a).$$

Here, $t = a$ lies within the range of integration. For other sets of limits, one needs to determine if the impulse lies within the range of integration. If it doesn't, then the integral is zero.

$$\int_{-\infty}^{\infty} \delta(t) e^{2t}\, dt = 1.$$
$$\int_{-\infty}^{\infty} \delta(t-1) e^{-t^2}\, dt = e^{-1}.$$
$$\int_{-\infty}^{\infty} \delta(t-2) \sin \pi t\, dt = \sin 2\pi t = 0.$$
$$\int_{0}^{\infty} \delta(t+2) t^3\, dt = 0.$$

The final answer is zero because the impulse is at $t = -2$ which outside the range of integration. So the impulse function is always zero within that range.

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7. Find the Laplace Transforms of the following functions:

(a) $e^{e^t}\delta(t-1)$, (b) $\displaystyle\sum_{n=0}^{\infty} \delta(t-n) = \delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots.$

[Look out for the geometric series....]

A7. (a) $\mathcal{L}[e^{e^t}\delta(t-1)] = \displaystyle\int_0^\infty e^{e^t}\delta(t-1)e^{-st}dt = e^{e^1}e^{-s} = e^{e-s}$ using the result for the integrals of delta functions.

(b) $\mathcal{L}[\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots]$.

As $\mathcal{L}[\delta(t-n)] = e^{-ns}$, this transform is $e^0 + e^{-s} + e^{-2s} + e^{-3s} + \ldots.$

Another way of writing this is $\displaystyle\sum_{n=0}^{\infty} e^{-ns}$, which is a geometrical series and it may be summed to get $1/(1-e^{-s})$.

The following summation of unit impulses, $\displaystyle\sum_{n=-\infty}^{\infty} \delta(t-n)$, (noting the lower limit) is known as the Shah function ($\text{Ш}(t)$) or, more descriptively, as the Dirac comb. It is used in signal processing and sampling. Question 18 on the next sheet concerns this function.

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8. Use the Laplace Transform to solve the following equations:

$$\text{(a)} \quad \frac{dy}{dt} + 3y = \delta(t), \qquad y(0) = 1.$$

$$\text{(b)} \quad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \delta(t), \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = b, \qquad \text{where } b \text{ is a constant.}$$

$$\text{(c)} \quad \frac{d^3y}{dt^3} - \frac{dy}{dt} = 3\delta(t), \qquad y(0) = 1, \quad \frac{dy}{dt}(0) = 0, \quad y''(0) = -1.$$

A8. (a) $\dfrac{dy}{dt} + 3y = \delta(t), \qquad y(0) = 1.$

The Laplace Transform of the full ODE is

$$(s+3)Y - 1 = 1,$$

and hence,

$$Y = \frac{2}{s+3} \qquad \Longrightarrow \qquad y = 2e^{-3t}.$$

Clearly the initial displacement given by the solution is not what was set as the initial condition, but this is due to the presence of the impulse.

(b) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = \delta(t), \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = b$ where b is a constant.

After taking the Laplace Transform the ODE becomes,

$$s^2Y - b - s + 3sY - 3 + 2Y = 1.$$

Hence,

$$(s^2 + 3s + 2)Y = s + 4 + b.$$

After rearrangement and then partial fractions we obtain,

$$Y = \frac{s+4+b}{(s+1)(s+2)} = \frac{3+b}{s+1} - \frac{2+b}{s+2},$$

upon using partial fractions. So the final solution is,

$$y = (3+b)e^{-t} - (2+b)e^{-2t}.$$

Although this solution yields $y(0) = 0$, as desired, it is easily shown that $y'(0) = 1 + b$. So a unit momentum has been added because of the presence of the unit impulse.

(c) $\dfrac{d^3y}{dt^3} - \dfrac{dy}{dt} = 3\delta(t), \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 0, \quad y''(0) = -1.$

After the taking of Laplace Transforms we get,

$$s^3Y - sY + 2 - s^2 = 3 \qquad \Longrightarrow \qquad Y = \frac{s^2+1}{s^3-s} \qquad \Longrightarrow \qquad Y = \frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{s}.$$

Hence the final solution is,

$$y = e^t + e^{-t} - 1.$$

If we check the initial conditions, then we see that $y(0) = 1$ and $y'(0) = 0$, as required. However, $y''(0) = 2$, according to the solution, whereas we imposed $y''(0) = -1$ at the start. We need to bear in mind that the $3\delta(t)$ forcing term for this 3rd order ODE increases the value of $y''(0)$ by 3 immediately, and this is what we have seen.

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9. Laplace Transforms are perfectly set up to solve Initial Value Problems, but let us try them out on a Boundary Value Problem. The aim, then, is to solve $y'' + y = 0$, subject to $y(0) = 1$ and $y(\frac{1}{2}\pi) = 1$. At the outset, let $y'(0) = c$ and carry out the analysis using this unknown constant. Eventually you will have the opportunity to find c.

A9. Setting $y'(0) = c$, the Laplace Transform of the ODE yields,

$$(s^2 + 1)Y - s - c = 0.$$

Hence,

$$Y = \frac{s + c}{s^2 + 1} = \frac{s}{s^2 + 1} + c\frac{1}{s^2 + 1}.$$

Hence,

$$y = \cos t + c \sin t.$$

Now we are in a position to satisfy the second Boundary Condition, $y(\frac{1}{2}\pi) = 1$; hence $c = 1$. The final solution is,

$$y = \cos t + \sin t.$$

Comment: So clearly it is possible to solve BVPs using Laplace Transforms. If one wishes to solve an equation or system of equations with n unknown initial conditions, then we will need to carry n unknown initial values at the start of the analysis. Satisfaction of the 'final' conditions would then yield n equations in n unknowns. This will inevitably involve matrix methods to find those unknowns.
