Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Sheet 11 — Laplace Transforms

10. First sketch the following functions, and then Find their Laplace Transforms:

(a) $H(t-a)t^3$ (b) $\sum_{n=0}^{\infty}(-1)^n H(t-n) = H(t) - H(t-1) + H(t-2) - H(t-3) + \ldots$

(c) $H(a-t)$,

[In one case it may be possible to simplify the final answer...]

A10. We are interested only in $t \geq 0$ because we are finding Laplace Transforms.

(a) This function looks like t^3 but that part to the left of $t = a$ is replaced by zero.

$$\mathcal{L}[H(t-a)t^3] = \int_0^\infty H(t-a)t^3 e^{-st}dt \qquad \text{by definition}$$

$$= \int_0^a 0\,dt + \int_a^\infty t^3 e^{-st}dt \qquad \text{splitting the range of integration}$$

$$= \frac{e^{-as}}{s^4}\Big[(as)^3 + 3(as)^2 + 6(as) + 6\Big] \qquad \text{after integration by parts.}$$

Note that, when $a = 0$, the result becomes $6/s^4$ which is $\mathcal{L}[t^3]$. Perhaps this shouldn't be surprising?

(b) This function looks like the classic crenellated walls of a castle.

$$\mathcal{L}[H(t) - H(t-1) + H(t-2) - H(t-3) + \ldots] = [1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} + \ldots]/s.$$

The solution may be rewritten in the more compact form,

$$\frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

and this geometric series may be summed to get

$$s^{-1}(1 + e^{-s})^{-1}.$$

(c) This function is equal to 1 when $t < a$ but is zero otherwise.

$$\mathcal{L}[H(a-t)] = \int_0^\infty H(a-t)\,e^{-st}\,dt = \int_0^a 1\,e^{-st}\,dt = \frac{1 - e^{-as}}{s}.$$

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11. Use the s-shift theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{1}{s+a}$ (b) $\dfrac{2}{(s+a)^3}$ (c) $\dfrac{b}{(s+a)^2+b^2}$ (d) $\dfrac{s}{(s+a)^2+b^2}$.

A11. The inverse Laplace Transform is denoted by $\mathcal{L}^{-1}[F(s)] = f(t)$.

The shift theorem used here is that $\mathcal{L}[e^{-at}f(t)] = F(s+a)$.

Hence we will be applying, $\mathcal{L}^{-1}[F(s+a)] = e^{-at}f(t)$.

(a) Given $\mathcal{L}[1] = 1/s$, we deduce that $\mathcal{L}[e^{-at}] = 1/(s+a)$ and hence that $e^{-at} = \mathcal{L}^{-1}[1/(s+a)]$.

(b) As $\mathcal{L}[t^2] = 2/s^3$, then $\mathcal{L}^{-1}[2/(s+a)^3] = t^2e^{-at}$.

(c) As $\mathcal{L}[\sin bt] = b/(s^2+b^2)$, then $\mathcal{L}^{-1}[b/((s+a)^2+b^2)] = e^{-at}\sin bt$.

(d) This one a little more complicated. We have an $(s+a)$ in the denominator, but not in the numerator. If we are to use the shift theorem, then it is necessary to have $(s+a)$ in all terms. Therefore we use

$$\frac{s}{(s+a)^2+b^2} = \frac{(s+a)-a}{(s+a)^2+b^2} = \frac{s+a}{(s+a)^2+b^2} - \frac{a}{b}\Big(\frac{b}{(s+a)^2+b^2}\Big).$$

The final sneaky move was based on the fact that $\mathcal{L}[\sin bt] = b/(s^2+b^2)$. Therefore we get the result that,

$$\mathcal{L}^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] = e^{-at}\Big[\cos bt - \frac{a}{b}\sin bt\Big].$$

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12. [This is an exam-style question.] Find the Laplace Transforms of both $\cos bt$ and $\sin bt$. Then use the s-shift theorem to write down the Laplace Transforms of $e^{-at}\cos bt$ and $e^{-at}\sin bt$. Hence solve the ODE,

$$y'' + 6y' + 25y = 0$$

subject to $y(0) = 1$ and $y'(0) = 5$.

A12. The ODE is

$$y'' + 6y' + 25y = 0$$

subject to $y(0) = 1$ and $y'(0) = 5$.

The ODE transforms into,

$$s^2Y - 5 - s + 6sY - 6 + 25Y = 0,$$

and so,

$$\begin{aligned} Y &= \frac{s+11}{s^2+6s+25} \\ &= \frac{s+11}{(s+3)^2+16} \\ &= \frac{s+3+8}{(s+3)^2+4^2} \qquad \text{now to use the earlier results} \\ &= \frac{s+3}{(s+3)^2+4^2} + 2\frac{4}{(s+3)^2+4^2} \\ \Longrightarrow \quad y &= e^{-3t}\cos 4t + 2e^{-3t}\sin 4t. \end{aligned}$$

So the tidied-up solution is

$$y = e^{-3t}\Big[\cos 4t + 2\sin 4t\Big].$$

This solution mat be checked to show that it does satisfy the given initial conditions.

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13. Use the t-Shift Theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{e^{-as}}{s^3}$ (b) $\dfrac{e^{-as}}{s+b}$ (c) $\dfrac{e^{-as}}{s^2+b^2}$ (d) $\dfrac{e^{-as}}{(s+c)^2+b^2}$.

A13. Here we use the result, $\mathcal{L}^{-1}[F(s)] = f(t) \quad \Rightarrow \quad \mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)$.

(a) $\mathcal{L}^{-1}\left[\dfrac{1}{s^3}\right] = \frac{1}{2}t^2 \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\dfrac{e^{-as}}{s^3}\right] = \frac{1}{2}H(t-a)(t-a)^2.$

(b) Answer is $H(t-a)e^{-b(t-a)}$.

(c) Answer is $\frac{1}{b}H(t-a)\sin b(t-a)$.

(d) Answer is $\frac{1}{b}H(t-a)e^{-c(t-a)}\sin b(t-a)$. Here we have used the answer to question 11c.

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14. Use the convolution theorem to find the Inverse Laplace Transform of:

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{1}{(s+a)(s^2+b^2)}$ (c) $\dfrac{1}{(s^2+a^2)^2}$ (d) $e^{-as}\times\dfrac{1}{s^2}$.

A14. The result we use here is that $\mathcal{L}[f*g]=F(s)G(s)$ where

$$f*g=\int_0^t f(\tau)g(t-\tau)d\tau \quad \text{or} \quad f*g=\int_0^t f(t-\tau)g(\tau)d\tau.$$

(a) Using $\mathcal{L}^{-1}\left[\dfrac{1}{s+a}\right]=e^{-at}$ we find that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right]=e^{-at}*e^{-at}=\int_0^t e^{-a\tau}e^{-a(t-\tau)}d\tau=\int_0^t e^{-at}d\tau=e^{-at}\int_0^t 1\,d\tau=te^{-at}.$$

(b) Using $\mathcal{L}^{-1}\left[\dfrac{1}{s+a}\right]=e^{-at}$ and $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+b^2}\right]=b^{-1}\sin bt$, we find that

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)(s^2+b^2)}\right]=e^{-at}*(b^{-1}\sin bt)=\frac{1}{b}\int_0^t e^{-a(t-\tau)}\sin b\tau d\tau$$

$$=\frac{e^{-at}}{b}\int_0^t e^{a\tau}\sin b\tau d\tau=\frac{1}{b(b^2+a^2)}\Big[be^{-at}+a\sin bt-b\cos bt\Big].$$

There is some integration by parts right at the very end.

(c) The solution here is that $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]=\dfrac{1}{a^2}\Big(\sin at\Big)*\Big(\sin at\Big)$ which may be shown, after some lengthy algebra, to reduce to

$$\frac{1}{a^2}\int_0^\tau \sin a\tau\sin a(t-\tau)d\tau=[\sin at-at\cos at]/2a^3.$$

(d) Don't be put off by the form of the question. The two functions we need are e^{-as}/s and $1/s$. Therefore we will set $F(s)=e^{-as}/s$ and $G(s)=1/s$. This means that $f(t)=H(t-a)$ and $g(t)=1$. Therefore the inverse Laplace Transform of e^{-as}/s^2 is,

$$f*g=\int_0^t f(\tau)g(t-\tau)\,d\tau \qquad \text{by definition}$$

$$=\int_0^t H(\tau-a)\times 1\,d\tau=\int_0^t H(\tau-a)\,d\tau$$

$$=\begin{cases}0 & (t<a)\\ t-a & (t>a)\end{cases}=(t-a)\times H(t-a).$$

That was one way of doing it which I think is the easier way. The other way is to recognise that $\mathcal{L}[\delta(t-a)]=e^{-as}$ and $\mathcal{L}[t]=1/s^2$. Therefore we set, $f(t)=\delta(t-a)$ and $g(t)=t$. Hence,

$$f*g=\int_0^t f(\tau)g(t-\tau)\,d\tau=\int_0^t \delta(\tau-a)\times(t-\tau)\,d\tau=\begin{cases}0 & (t<a)\\ t-a & (t>a)\end{cases}=(t-a)\times H(t-a).$$

Do bear in mind that, in both the integrals above, $H(\tau - a) = 0$ and $\delta(\tau - a) = 0$ when $t < a$, and therefore the integrals are zero in such cases.

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15. Use the convolution theorem to find the solutions to the following equations:

(a) $y' + 3y = e^{-2t}, \quad y(0) = 0$;

(b) $y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 1$;

(c) $y'' + y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$.

A15. (a) The standard procedure for solving such equations leads first to,

$$(s+3)Y = \frac{1}{s+2},$$

and therefore,

$$Y = \frac{1}{(s+2)(s+3)}.$$

This could be inverted by first using partial fractions, but we will use the convolution theorem instead. The definition of the convolution is given by the red text in the solutions to Q14, and the theorem may be stated in the form,

$$\mathcal{L}[f * g] = F(s)G(s),$$

or, for the purposes of this question, as,

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g.$$

If we let $f = e^{-2t}$ and $g = e^{-3t}$, we have $F = 1/(s+2)$ and $G = 1/(s+3)$. Therefore,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\frac{1}{(s+3)}\right] &= e^{-2t} * e^{-3t} \\
&= \int_0^t e^{-2\tau} e^{-3(t-\tau)}\, d\tau \\
&= e^{-3t}\int_0^t e^{-2\tau} e^{3\tau}\, d\tau \\
&= e^{-3t}\int_0^t e^{\tau}\, d\tau \\
&= e^{-3t}\left[e^t - 1\right] \\
&= e^{-2t} - e^{-3t}.
\end{aligned}$$

We could, of course, have changed the order of the functions, e^{-2t} and e^{-3t}, in the convolution integral without changing either the answer or the degree of difficulty of the calculations.

(b) Again, we get,

$$s^2Y - 1 + 5sY + 6Y = 0 \qquad \Rightarrow \qquad Y = \frac{1}{(s+2)(s+3)},$$

which is actually precisely identical to the solution in part (a).

(c) A little more tricky.... We get,

$$(s^2+1)Y = \frac{1}{s+1}.$$

Therefore,

$$Y = \frac{1}{(s^2+1)(s+1)}.$$

Once more, partial fractions could be used, but this is a product of two functions of s, both of whose inverse transforms we know. So we let, $f = \sin t$ and $g = e^{-t}$, and so $F = 1/(s^2+1)$ and $G = 1/(s+1)$. The convolution theorem now gives,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s+1)}\right] &= \sin t * e^{-t} \\
&= \int_0^t \sin\tau\, e^{-(t-\tau)}\, d\tau \\
&= e^{-t}\int_0^t e^{\tau}\, \sin\tau\, d\tau \\
&= e^{-t}\,\tfrac{1}{2}\left[e^{\tau}(\sin\tau-\cos\tau)\right]_0^t \qquad \text{integration by parts} \\
&= \tfrac{1}{2}e^{-t}\left[e^{t}(\sin t-\cos t)-(-1)\right] \\
&= \tfrac{1}{2}\left[\sin t-\cos t+e^{-t}\right].
\end{aligned}$$

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16. Solve the system of equations,

$$x' = 2x - 4y + \delta(t),$$
$$y' = 3x - 5y,$$

subject to the initial conditions, $x(0) = y(0) = 0$.

A16. Finally, a question involving both a system and a unit impulse! Something new...

All the usual tricks give us,

$$(s-2)X = -4Y + 1, \qquad (s+5)Y = 3X.$$

Eliminate X between these to get,

$$(s^2+3s+2)Y = 3.$$

Rearranging and then the use of partial fractions gives,

$$Y = \frac{3}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{3}{s+2}.$$

Inversion gives,

$$y = 3\Big(e^{-t} - e^{-2t}\Big).$$

Substitution of this into the second of the original equations gives,

$$x = 4e^{-t} - 3e^{-2t}.$$

If we check to see if the *given* initial conditions have been satisfied, then find that $y(0) = 0$, as given. But we have $x(0) = 1$, and therefore the presence of the unit impulse in the x-equation, which has a single derivative with a unit coefficient, causes the initial value of x to change discontinuously by a unit amount.

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17. Solve the system of equations,

$$x'' + 2x - 2y = \delta(t),$$
$$y'' - x + 3y = 0,$$

subject to the initial conditions, $x(0) = x'(0) = y(0) = y'(0) = 0$. When the final solution has been obtained, determine which of the four initial conditions has been violated, but can you guess this in advance?

A17. Given the initial conditions, $\mathcal{L}[x''] = s^2X$, with the same type of result for y''. Hence the above equations transform to,

$$(s^2 + 2)X - 2Y = 1, \qquad (s^2 + 3)Y - X = 0. \tag{1}$$

As in Q16 we may eliminate either X to get an algebraic equation for Y, or eliminate Y to get an algebraic equation for X. At the outset there is nothing to choose between these two routes, but the former turns out to be slightly easier. Therefore we get

$$(s^4 + 5s^2 + 4)Y = 1,$$

and therefore,

$$Y = \frac{1}{s^4 + 5s^2 + 4} = \frac{1}{(s^2 + 1)(s^2 + 4)}.$$

Use of partial fractions gives,

$$Y = \frac{1/3}{s^2 + 1} - \frac{1/3}{s^2 + 4},$$

and hence,

$$y = \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t.$$

We could either substitute this expression for y into the ODE in the question which invols y'' to find x. Or we sustitute for Y into Eq. (1) to find X, and then find its inverse Laplace Transform. A third route would be to eliminate Y between the two equations in (1). I'll opt randomly for the third route; we get,

$$X = \frac{s^2 + 3}{s^4 + 5s^2 + 4},$$

which may be simplified using partial fractions:

$$X = \frac{2/3}{s^2 + 1} + \frac{1/3}{s^2 + 4}.$$

Hence,

$$x = \tfrac{2}{3}\sin t + \tfrac{1}{6}\sin 2t.$$

Clearly the initial conditions, $x(0) = 0$ and $y(0) = 0$ are satisfied because of the sine terms. However, we find that, while $y'(0) = 0$, as imposed initially, we have $x'(0) = 1$. This violation of the given initial conditions is caused again by the presence of the unit impulse in the equations for x where the second derivative has a unit coefficient.

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18. [This is a project-like question which combines quite a large number of mathematical results.]

The overall aim for this question is to solve the ODE,

$$y' + y = \text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

The unusual symbol, Ш, which I cannot typeset properly(!), is known as the Shah function, and the symbol itself is the Cyrillic character, sha, which mimics the shape of the function. In various contexts it is also known as (i) the Dirac comb, (ii) more picturesquely as the bed of nails function, and (iii) more functionally as an impulse train.

The solution is a periodic function which has a period of 1, but this can't be found simply using the Laplace Transform because that is an integral from $t = 0$ onwards, whereas the Shah function consists of impulses at all integer values of t, both positive and negative. Therefore we will do this by determining the eventual 'steady periodic state' that is achieved when t becomes large. Enjoy the ride!

(a) Find the Laplace Transform of e^{-t}, and then apply the t-shift theorem to find the inverse Laplace Transform of $e^{-ns}/(s+1)$.

(b) Use the Laplace Transform on the ODE,

$$y' + y = \sum_{n=0}^{\infty} \delta(t-n), \qquad y(0) = 0,$$

to find an expression for $Y(s)$, the transform of $y(t)$. Do not simplify this expression for Y by, say, summing the geometric series!

(c) Now use the result of part (a) to write down an expression for $y(t)$ in terms of a sum of terms involving unit step functions.

(d) The sum we have obtained for $y(t)$ is infinite in length; do make sure that you're happy with this idea! Now we let $t = n + \epsilon$ in your expression for y, where n is the first positive integer below t, and where $0 \leq \epsilon < 1$. You should then be able to factor $e^{-\epsilon}$ out of the resulting mess(!), and then be able to sum the resulting geometric series to obtain a compact formula for y.

(e) Now find $\lim_{n\to\infty} y$. This will give the long-term formula for $y(t)$, but written in terms of ϵ — this will be a valid formula for $y(t)$ between two neighbouring integer values of t. In the ultimate steady periodic state, what are the maximum and minimum values of y?

(f) See if you can guess what $y(t)$ looks like.

(g) An easier way to solve the main problem is to concentrate on the interval of time, $0 \leq t < 1$, and to solve for

$$y' + y = \delta(t), \qquad y(0) = c,$$

using Laplace Transforms. The value of c may be found by insisting that $y(1) = y(0)$.

A18. (a) We have,

$$\mathcal{L}[e^{-t}] = \int_0^\infty e^{-t}\, e^{-st}\, dt = \frac{1}{s+1}.$$

Use of the t-shift theorem yields,

$$\mathcal{L}^{-1}\Big[\frac{e^{-ns}}{s+1}\Big] = H(t-n)e^{-(t-n)}.$$

(b) The Laplace Transform of the ODE is,

$$(s+1)Y = 1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots .$$

Although this is a geometric series, we have been warned against being tempted to sum this series. So the required solution for Y at this stage is,

$$Y = \frac{1}{s+1} + \frac{e^{-s}}{s+1} + \frac{e^{-2s}}{s+1} + \frac{e^{-3s}}{s+1} + \cdots .$$

(c) This yields,

$$y = H(t)e^{-t} + H(t-1)e^{-(t-1)} + H(t-2)e^{-(t-2)} + H(t-3)e^{-(t-3)} + \cdots$$

using the result of Q18a, above.

(d) If we have a value of t which lies between neighbouring integers, such as $n < t < n+1$, then the stepfunctions, $H(t-n-1)$, $H(t-n-2)$, $H(t-n-3)$ and so on are all zero. This leaves us with the finite series,

$$y = H(t)e^{-t} + H(t-1)e^{-(t-1)} + H(t-2)e^{-(t-2)} + \cdots + H(t-(n-1))e^{-(t-(n-1))} + H(t-n)e^{-(t-n)}.$$

Now let $t = n+\epsilon$, as asked, where ϵ plays the role of time between $t = n$ and $t = n+1$. Hence the solution for y at $t = n+\epsilon$ becomes,

$$\begin{aligned} y &= e^{-(n+\epsilon)} + e^{-(n-1+\epsilon)} + e^{-(n-2+\epsilon)} + \cdots + e^{-(2+\epsilon)} + e^{-(1+\epsilon)} + e^{-(\epsilon)} \\ &= e^{-\epsilon}\left[e^{-n} + e^{-(n-1)} + e^{-(n-2)} + \cdots + e^{-2} + e^{-1} + 1\right] \\ &= e^{-\epsilon}\left[1 + e^{-1} + e^{-2} + \cdots + e^{-(n-2)} + e^{-(n-1)} + e^{n}\right]. \qquad \text{back to front} \end{aligned}$$

This geometric series may be summed in the usual way to obtain,

$$y = e^{-\epsilon}\frac{[1 - e^{-(n+1)}]}{[1 - e^{-1}]}.$$

(e) We may find the ultimate periodic state of the solution by letting $n \to \infty$, and this yields,

$$y = e^{-\epsilon}\frac{1}{1 - e^{-1}}.$$

Recall that this is valid for $n < t < n+1$ where $t = n+\epsilon$.

This expressione for y is a decreasing function of ϵ, and therefore the respective maximum and minimum of this function take place as follows,

$$y(\epsilon = 0) = \frac{1}{1 - e^{-1}}, \qquad y(\epsilon = 1) = \frac{e^{-1}}{1 - e^{-1}}.$$

The difference between these values is 1, as one might expect because the next interval of time of length, 1, begins with the effect of the next unit impulse.

(f) I can't guess what you will guess! The function is given below, though.

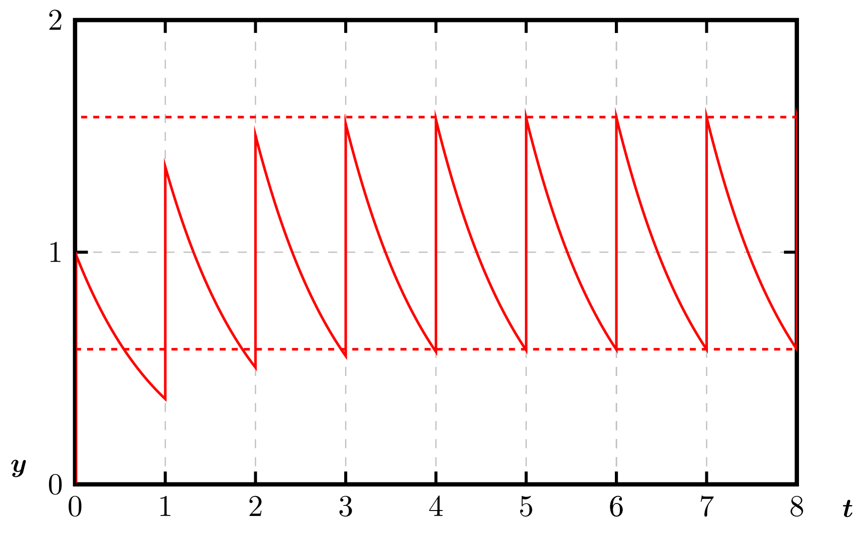
(g) So we will solve, $y' + y = \delta(t)$ subject to $y(0) = c$. The taking of the Laplace Transform gives,

$$(s+1)Y - c = 1 \quad \Longrightarrow \quad Y = \frac{1+c}{s+1} \quad \Longrightarrow \quad y = (1+c)e^{-t}.$$

If the interval from $t = 0$ to $t = 1$ is to represent one period (i.e. the solution of the original ODE at the start of the question, then we will require $y(1) = c$ as well. Hence

$$(1+c)e^{-1} = c \quad \Longrightarrow \quad c = \frac{1}{1 - e^{-1}}.$$

There is a sense in which this last result is a little depressing given how much work went into parts (a) to (f)!



The sudden vertical jumps (all of length, 1) are caused by the successive unit impulses. This graph also shows how quickly the transients decay to form what is the eventual periodic solution for the case when the Shah function is the forcing term.
