Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem sheet 9b — ODEs (extension)

1. One notation for dy/dt which is sometimes used in textbooks and research papers is Dy. In essence, d/dt and D are directly equivalent to one another and are simply alternative ways of writing down the same thing. Given this, one may try to determine the inverse of D in the following way. Given that

$$\frac{dy}{dt} = f(t) \qquad \Rightarrow \qquad y = c + \int f(t)\,dt,$$

then we may define D^{-1} as follows,

$$Dy = f \qquad \Rightarrow \qquad y = \frac{1}{D}f(t) \qquad = \qquad c + \int f(t)\,dt.$$

In other words, D^{-1} is equivalent to an indefinite integral plus an arbitrary constant.

(a) Now consider the differential equation, $(D+a)y = f(t)$. Rewrite this in the usual way (i.e. $dy/dt + ay = f(t)$) and use the integrating factor approach to find y, not forgetting the arbitrary constant. When this is done, identify which part of your solution forms the Complementary function and which the Particular Integral. What you have written is then the equivalent of

$$y = \frac{1}{D+a}f(t),$$

and it defines the meaning of $(D+a)^{-1}$.

(b) Let us extend the result of Q1a to the following differential equation,

$$\frac{d^2y}{dt^2} + (a+b)\frac{dy}{dt} + aby = f(t).$$

This may also be written as

$$D^2y + (a+b)Dy + aby = f(t), \qquad \text{or} \qquad (D+a)(D+b)y = f(t).$$

If we now set $z = (D+b)y$ then $(D+a)z = f(t)$.

First solve $(D+a)z = f(t)$ for z by applying the result of Q1a directly. Then solve $(D+b)y = z$ to find y. Keep your wits about you on this one — the final answer will involve a double integral.

(c) Now we will modify slightly the answer given in Q1b for the case when $a = b$, which (in the terminology of the lectures) is a repeated-λ case. You should find that some integrals will simplify slightly.

(d) Apply the formula found in Q1b to solve the two equations,

$$y'' + 3y' + 2y = e^t \qquad \text{and} \qquad y'' + 3y' + 2y = e^{-t}.$$

(e) Suppose that we are solving a third order ODE with $f(t)$ on the right hand side. If it is written in the form,

$$(D+a)(D+b)(D+c)y = f(t),$$

and given the form of the answer Q1b, can you guess what the solution is?

A1. (a) We are solving $(D+a)y = f(t)$ or, in the standard notation, $y' + ay = f(t)$. There is an invitation to employ an integration factor which, in this case, is

$$\text{I.F.} = \exp\left[\int a\,dt\right] = e^{at}.$$

Therefore

$$\frac{dy}{dt} + ay = f(t) \qquad \text{becomes} \qquad e^{at}\Big(\frac{dy}{dt} + ay\Big) = e^{at}f(t) \qquad \Rightarrow \qquad \frac{d}{dt}\Big(e^{at}y\Big) = e^{at}f(t).$$

The left hand side of this latest equation is an exact differential, and therefore we may integrate to obtain,

$$e^{at}y = c + \int e^{at}f(t)\,dt \qquad \Rightarrow \qquad y = c\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt.$$

Clearly, the first part of the solution is the Complementary Function, while the one involving the integral is the Particular Integral. Therefore,

$$\frac{1}{D+a}f(t) = c\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt.$$

(b) First we solve $(D+a)z = f$ using the result of Q1a. We have,

$$z = c_1\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt. \tag{1}$$

Now we solve for $(D+b)y = z$. We have,

$$y = c_2\,e^{-bt} + e^{-bt}\int e^{bt}z(t)\,dt. \tag{2}$$

Now we shall substitute the expression for z given in (1) into the solution for y in equation (2):

$$\begin{aligned}
y &= c_2\,e^{-bt} \quad + \quad e^{-bt}\int e^{bt}\left[c_1\,e^{-at} + e^{-at}\int e^{at}f(t)\,dt\right]\,dt \\
&= c_2\,e^{-bt} \quad + \quad c_1 e^{-bt}\int e^{(b-a)t}\,dt \quad + \quad e^{-bt}\int e^{(b-a)t}\left[\int e^{at}f(t)\,dt\right]\,dt \\
&= c_2\,e^{-bt} \quad + \quad c_1 e^{-bt}\left[\frac{e^{(b-a)t}}{b-a}\right] \quad + \quad e^{-bt}\int e^{(b-a)t}\left[\int e^{at}f(t)\,dt\right]\,dt \\
&= c_2\,e^{-bt} \quad + \quad c_1^* e^{-at} \quad + \quad e^{-bt}\int e^{(b-a)t}\left[\int e^{at}f(t)\,dt\right]\,dt, \qquad (3)
\end{aligned}$$

where c_1^* is a redefined arbitrary constant. Here, the terms involving c_1 and c_2, the arbitrary constants, form the Complementary Function, and the term with the double integral is the Particular Integral. This expression is correct for any choice of the constants, a and b, but only if they are different.

(c) This is not simply a case of replacing b by a in the last line of Equation (3), above. Rather, we need to rework the analysis which leads to (3) by beginning with the very first line of that equation, as follows.

$$\begin{aligned} y &= c_2\, e^{-at} + e^{-at}\int e^{at}\left[c_1\, e^{-at} + e^{-at}\int e^{at} f(t)\, dt\right] dt \\ &= c_2\, e^{-bt} + c_1 \int 1\, dt + e^{-at}\int \left[\int e^{at} f(t)\, dt\right] dt \\ &= c_2\, e^{-at} + c_1 t e^{-at} + e^{-at}\int\left[\int e^{at} f(t)\, dt\right] dt. \qquad (4) \end{aligned}$$

(d) For both equations we may use $a = 1$ and $b = 2$ in (3). For both equations we obtain Complementary functions of the form, Ae^{-2t} and Be^{-t}, where I have reverted to the notation used in the lectures for the arbitrary constants.

For the Particular Integral for the first equation we have,

$$\begin{aligned} y_{\rm pi} &= e^{-2t}\int e^t\Big[\int e^t \times e^t\, dt\Big]\, dt && \text{subst. into (3)} \\ &= e^{-2t}\int e^t\Big[\int e^{2t}\, dt\Big]\, dt && \text{on mutiplying} \\ &= e^{-2t}\int e^t\Big[\tfrac{1}{2}e^{2t}\Big]\, dt && \text{on integrating} \\ &= \tfrac{1}{2}e^{-2t}\int e^{3t}\, dt && \text{on multiplying} \\ &= \tfrac{1}{2}e^{-2t}\times\tfrac{1}{3}e^{3t} && \text{on integrating} \\ &= \tfrac{1}{6}e^t. \end{aligned}$$

Hence the full solution for the first equation is,

$$y = Ae^{-2t} + Be^{-t} + \tfrac{1}{6}e^t.$$

For the second ODE we have,

$$\begin{aligned} y_{\rm pi} &= e^{-2t}\int e^t\Big[\int e^t \times e^{-t}\, dt\Big]\, dt && \text{subst. into (3)} \\ &= e^{-2t}\int e^t\Big[\int 1\, dt\Big]\, dt \\ &= e^{-2t}\int e^t\, t\, dt \\ &= e^{-2t}\Big[(t-1)e^t\Big] \\ &= (t-1)e^{-t}. \end{aligned}$$

If we had solved this equation using the standard CF/PI approach, then the PI would have been just te^{-t}, and this would also have been true if we had used $a = 2$ and $b = 1$ above — do check this outrageous statement!

In any case, the extra $-e^{-t}$ may be swallowed up in the CF, given that part of the CF is Be^{-t} where B is arbitrary.

The final solution is,

$$y = Ae^{-2t} + Be^{-t} + te^{-t}.$$

(e) The solution is,

$$y = c_1e^{-at} + c_2e^{-bt} + c_3e^{-ct} + e^{-ct}\int e^{(c-b)t}\left[\int e^{(b-a)t}\left[\int e^{at}f(t)\,dt\right]dt\right]dt.$$

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2. The aim for this question is to solve $y' + ay = 1$ subject to $y(0) = 0$ using Taylor's series. First, write down a general expression for the Taylor's series about $t = 0$ for the function $y(t)$ — this is *not* the solution because we don't yet know the value of all of the derivatives of y at $t = 0$. However, we may substitute the initial value of y into the governing equation to find $y'(0)$. Now differentiate the governing equation once; this will allow us to find $y''(0)$. Differentate again and hence find $y'''(0)$. The pattern should now be clear. Hence write down the Taylor's series of the solution. Can you identify it?

A2. The required Taylor's series is

$$y(t) = y(0) + \frac{y'(0)}{1}t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \cdots.$$

Successive derivatives of the governing equation, $y' + ay = 1$, are,

$$y'' + ay' = 0, \qquad y''' + ay'' = 0, \qquad y^{(4)} + ay''' = 0, \qquad y^{(5)} + ay^{(4)} = 0, \cdots.$$

If $y(0) = 0$, then the ODE gives us that $y'(0) = 1$.

The derivative of the ODE now tells us that $y''(0) = -ay'(0) = -a$.

The next derivative yields $y'''(0) = +a^2$ and so on with $-a^3$ and $+a^4$ for the next two derivatives at $t = 0$. Hence the solution may be written as,

$$y = t - \frac{at^2}{2!} + \frac{a^2t^3}{3!} - \frac{a^3t^4}{4!} + \frac{a^4t^5}{5!} + \cdots.$$

We know that

$$e^{-at} = 1 - at + \frac{a^2t^2}{2!} - \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \cdots,$$

and therefore our Taylor's series solution represents,

$$y = \frac{1 - e^{-at}}{a}.$$

Yes, that took a few lines of working-out and a bit of head-scratching....

3. This question was devised while I was watching the film, Gravity, en route to India, with only a thin skin of aluminium between me and a quarter of an atmosphere of air pressure at -50°C while travelling at 500mph six miles above the ground. I am not sure that I like disaster movies while flying!

Suppose that Sandra Bullock and George Clooney are stranded in space, 20m apart and stationary relative to one another. We wish to determine how long will it take for gravitational attraction to cause the couple get close enough together that they may grasp each other's hand?

We will use $x(t)$ to denote their distance from their mutual centre of gravity. The governing equation, a nonlinear second order equation, is then,

$$m_1\frac{d^2x}{dt^2} = -\frac{m_1m_2G}{4x^2}.$$

Here, $m_1 = m_2 = 60$kg are their masses (yes, I shall assume that they have the same mass!) and $G = 6.67408 \times 10^{-11}$N m^{2}kg^{-2} is the gravitational constant. At the initial time, $t = 0$, their separation is $20\mathrm{m}$, i.e. the $x = 10\mathrm{m}$, and our aim is to find when gravitational attraction draws them close enough to touch fingers, say, $x = ½\mathrm{m}$.

(a) Nothing in our lectures hints about how to solve a nonlinear second order ODE! However, d^2x/dt^2 is the same as dv/dt where $v = dx/dt$. Use the chain rule to show that

$$\frac{dv}{dt} = v\frac{dv}{dx}.$$

Use this substitution to solve for v in terms of x. Apply the initial condition, namely that $x = x_0$ and $v = 0$ at $t = 0$ (we'll keep the initial separation general for now, although it is clear that $x_0 = 10$).

(b) Now that we have v in terms of x, it is possible to solve this by first using the substitution, $x = x_0\cos^2\theta$, to obtain an equation for θ in terms of t. This equation may be solved to find t in terms of θ. Don't let this worry you, for the whole point is that you need to find the time corresponding to a given distance (for which θ is a proxy).

Now use $x_0 = 10$ and let $x = ½$ in your solution (hint: it's probably better to find the corresponding value of θ here) to find the corresponding time. So how many days does it take for them to be reunited? (Cue suitable sad violin music...)

A3. Now, if you are presently scratching your head wondering why I have made a mistake in writing down that equation for gravitational attraction, then allow me to explain a little about why it is correct in this context. I am addressing the presence of the 4 in the denominator which tends not to be present when one checks it out on Wikipedia, for example. A very simplistic reason is that the x in the denominator is assumed to be the distance between the two masses isn't x here, but is $2x$. While this seems reasonable, that argument isn't perfectly rigorous. I am happy to provide a more detailed reasoning upon request. So we are taking

$$m_1\frac{d^2x}{dt^2} = -\frac{m_1m_2G}{(2x)^2},$$

as the equation of motion where x is the distance from the origin, rather than the distance between two masses.

(a) We know that

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dv}{dt},$$

and we wish to change the t-derivative to an x-derivative. The chain rule gives us,

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}.$$

Hence the governing equation becomes,

$$v\frac{dv}{dx} = -\frac{Gm}{4x^2},$$

which is of variables-separable form, and where I have cancelled the m_1 coefficients on both sides and where m_2 will be written as m from now on. On separating the variables we have,

$$v\,dv = -\frac{Gm}{4x^2}\,dx,$$

which, upon integration yields,

$$\tfrac{1}{2}v^2 = \frac{Gm}{4x} + c.$$

When $t = 0$ we have $x = x_0 = 10$ and $v = 0$, and therefore $c = -Gm/4x_0$. Therefore the solution so far is,

$$v^2 = \frac{Gm}{2}\left(\frac{1}{x} - \frac{1}{x_0}\right),$$

and hence

$$v = -\sqrt{\frac{Gm}{2}}\left(\frac{1}{x} - \frac{1}{x_0}\right)^{1/2}.$$

When taking the square roots of v^2, the negative sign has been taken because x will decrease in time and hence $v < 0$.

(b) Now we revert to x, and therefore the latest equation becomes,

$$\frac{dx}{dt} = -\sqrt{\frac{Gm}{2}}\left(\frac{1}{x} - \frac{1}{x_0}\right)^{1/2}. \qquad (1)$$

Following the hint, we will change variable from x to θ using $x = x_0\cos^2\theta$. So

$$\frac{dx}{dt} = -2x_0\sin\theta\cos\theta\,\frac{d\theta}{dt}.$$

Hence the equation becomes,

$$\begin{aligned} -2x_0\sin\theta\cos\theta\,\frac{d\theta}{dt} &= -\frac{\sqrt{Gm}}{\sqrt{2x_0}}\left[\frac{1}{\cos^2\theta} - 1\right]^{1/2} \\ &= -\frac{\sqrt{Gm}}{\sqrt{2x_0}}\left[\frac{1-\cos^2\theta}{\cos^2\theta}\right]^{1/2} \\ &= -\frac{\sqrt{Gm}}{\sqrt{2x_0}}\frac{\sin\theta}{\cos\theta}. \end{aligned}$$

Therefore the equation may be tidied up to give,

$$\cos^2\theta\frac{d\theta}{dt} = \frac{\sqrt{Gm}}{2\sqrt{2}\,x_0^{3/2}},$$

or, even better,

$$\tfrac{1}{2}(1+\cos 2\theta)\frac{d\theta}{dt} = \frac{\sqrt{Gm}}{2\sqrt{2}\,x_0^{3/2}}.$$

This is another separation-of-variables equation, and we may integrate to get,

$$\tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) = \frac{\sqrt{Gm}}{2\sqrt{2}\,x_0^{3/2}}t + c.$$

The initial condition is that, at $t=0$ we have $x=x_0$ and hence $\cos\theta = 1$ or $\theta = 0$. Therefore $c=0$. Therefore we may now write the final solution (in terms of θ) as

$$t = 2\left(\frac{x_0^3}{2Gm_2}\right)^{1/2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right).$$

Although we cannot rearrange this equation to get θ (and hence x) in terms of t, it doesn't matter here, for we need to find t when $x=0.5$. This translates into when $\cos^2\theta = 0.5/10$, i.e. $\theta = 1.34528$ radians. Hence

$$t = 2\times 1.563228\times\left(\frac{1000}{2\times 60\times 6.67408\times 10^{-11}}\right)^{1/2} = 1\,194\,755s.$$

This is just under 12 days and 19 hours. At that point in time their relative speed would be about 0.123mm/s (using Eq. (1)), or just over the width of the human hair per second, so it would be a very gentle meeting of fingertips. Nice.

However, given that the human body can survive at most only three days without food and water, this tale has a very sad ending....no violin music....not that it can be heard in space....

If this had been a purely theoretical problem involving point masses instead of film stars, then the point masses would have collided only about 89 minutes later, when $\theta = \frac{1}{2}\pi$ and at an infinite velocity!

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4. The Cauchy-Euler equation is a different class of linear ODE, and technically it is known as an equi-dimensional equation. The most general second order version is

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

There are two ways of solving this equation, the first being to let $y = x^n$ (and then one will eventually be led to an indicial/auxiliary/characteristic equation for n) while the second is to change variables from x to ξ using $x = e^\xi$.

(a) Try to solve the equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

using each of these two methods. [Note, when attempting the second, we are changing from dy/dx to $dy/d\xi$, and the chain rule will need to be used. Take care with the transformation of the second derivative — the product rule will be needed!]

(b) Suppose now that we wish to solve

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0.$$

The first method given above leads to a repeated value of n and then it isn't obvious how to proceed in this context. So adopt the second method, solve the equation, and this will show how one should proceed when using the otherwise quicker and simpler first method.

A4. For $x^2y'' + 4xy' + 2y = 0$, we let $y = x^n$ as suggested. This yields,

$$x^2\Big[n(n-1)x^{n-2}\Big] + 4x\Big[nx^{n-1}\Big] + 2x^n = 0$$

$$\Longrightarrow \qquad \Big[(n^2 - n) + 4n + 2\Big]x^n = 0.$$

$$\Longrightarrow \qquad \Big[n^2 + 3n + 2\Big]x^n = 0.$$

This shows why the given substitution works: the only function which, when it is differentiated m times and then multiplied by x^m, gives the same function is a power of x. The auxiliary equation for n is now

$$n^2 + 3n + 2 = 0,$$

and we find that $n = -1, -2$. Hence the solution follows in the same way as we have for constant-coefficient equations:

$$y = Ax^{-1} + Bx^{-2}. \tag{4}$$

NOTE: the Cauchy-Euler ODEs uually arise when considering problems in polar coordinates. This will not happen in Year 1.

Although the above is sufficient in and of itself, there is the following alternative route. This will be typeset in dark orange.

We may substitute $x = e^\xi$, then we need to change from x-derivatives to ξ-derivatives. Therefore,

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = e^{-\xi}\frac{dy}{d\xi}.$$

This is more easily dealt with if we multiply both sides (i.e. leftmost and rightmost) by e^ξ (i.e. x). We get,

$$x\frac{dy}{dx} = \frac{dy}{d\xi},$$

which is very useful. For the second derivatives, note first that,

$$\frac{d^2y}{d\xi^2} = \frac{d}{d\xi}\Big(\frac{dy}{d\xi}\Big) = x\frac{d}{dx}\Big(x\frac{dy}{dx}\Big) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}.$$

Hence,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}.$$

Therefore the given equation transforms as follows,

$$\Big(\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\Big) + 4\frac{dy}{d\xi} + 2y = 0,$$

and hence

$$\frac{d^2y}{d\xi^2} + 3\frac{dy}{d\xi} + 2y = 0.$$

Therefore our substitution transforms a Cauchy-Euler equation into a constant-coefficient equation. Also nice, and also extremely useful as we will see. The solution of this latest equation is,

$$y = Ae^{-\xi} + Be^{-2\xi} = Ax^{-1} + Bx^{-2},$$

which is identical to equation (4), above.

(b) When we consider the final equation in the question, $x^2y'' + 5xy' + 4y = 0$, the $y = x^n$ trial solution yields the repeated pair, $n = -2, -2$. Therefore let us see what the coordinate transformation method gives us. Using the above results we obtain,

$$\frac{d^2y}{d\xi^2} + 4\frac{dy}{d\xi} + 4y = 0.$$

Letting $y = e^{\lambda\xi}$ also yields $\lambda = -2, -2$, and therefore the solution is

$$y = (A + B\xi)e^{-2\xi}.$$

When we revert to x, this becomes,

$$y = (A + B\ln x)x^{-2}.$$

This gives us the clue for how one progresses quickly when repeated roots of the auxiliary equation are encountered when solving a Cauchy-Euler equation.