Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem sheet 6 — Series – Binomial Series

1. Find the binomial expansions for **(a)** $(1+x)^{-2}$ and **(b)** $(1+x)^{-3}$, and express the final solution in summation form. Check your result for part (b) by differentiating the result for part (a). [You may need to alter the summation variable to get a perfect match.]

A1. We will use the standard binomial expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots . \qquad \text{(Q1)}$$

(a) We use $n=-2$ in Eq. (Q1). We get,

$$\begin{aligned}(1+x)^{-2} &= 1 + (-2)x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{2.3}x^3 + \frac{(-2)(-3)(-4)(-5)}{2.3.4}x^4 \\ &\qquad + \frac{(-2)(-3)(-4)(-5)(-6)}{2.3.4.5}x^5 + \cdots \\ &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n.\end{aligned}$$

(b) We use $n=-3$ in Eq. (Q1) above. We get,

$$\begin{aligned}(1+x)^{-3} &= 1 + (-3)x + \frac{(-3)(-4)}{2}x^2 + \frac{(-3)(-4)(-5)}{2.3}x^3 + \frac{(-3)(-4)(-5)(-6)}{2.3.4}x^4 \\ &\qquad + \frac{(-3)(-4)(-5)(-6)(-7)}{2.3.4.5}x^5 + \cdots \\ &= 1 - 3x + \frac{3\cdot 4}{2}x^2 - \frac{4\cdot 5}{2}x^3 + \frac{5\cdot 6}{2}x^4 - \frac{6\cdot 7}{2}x^5 + \cdots \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)x^n.\end{aligned}$$

It is worth noting that the manner in which I have written down the series on the penultimate line came from looking at the later terms, not the first few. The few were then checked to see if they fitted into the general formula, and indeed they do.

Checking part (b) using the solution to part (a).

In part (a) we had,

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots ,$$

and the derivative if this expression yields,

$$-2(1+x)^{-3} = -2 + 2\cdot 3x - 3\cdot 4x^2 + 4\cdot 5x^3 - 5\cdot 6x^4 + \cdots,$$

and hence,

$$(1+x)^{-3} = 1 - 3x + \frac{3\cdot 4}{2}x^2 - \frac{4\cdot 5}{2}x^3 + \frac{5\cdot 6}{2}x^4 + \cdots,$$

which is fine.

We may also check the summation form of the solutions. We had,

$$(1+x)^{-2} = \sum_{n=0}^{\infty} (-1)^n (n+1)x^n. \tag{1}$$

When differentiated we get,

$$-2(1+x)^{-3} = \sum_{n=0}^{\infty} (-1)^n n(n+1)x^{n-1}. \tag{2}$$

This expression has a problem: when $n = 0$ we have x^{-1} as the power of x, but at least it is multiplied by n. In fact, this term came from differentiating a constant, and therefore it will be zero anyway. But it will be better to remove the $n = 0$ term from the summation. Hence Eq. (2) becomes,

$$-2(1+x)^{-3} = \sum_{n=1}^{\infty} (-1)^n n(n+1)x^{n-1}.$$

Now we'll divide by -2 to obtain,

$$(1+x)^{-3} = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n-1} n(n+1)x^{n-1}. \tag{3}$$

This is correct, but it doesn't look like our previous solution, and therefore we need to change the summation counter so that it begins at zero. This is a little like integration-by-substitution! We'll let $n = m + 1$, and therefore Eq. (3) becomes,

$$(1+x)^{-3} = \frac{1}{2}\sum_{m=0}^{\infty} (-1)^m (m+1)(m+2)x^m. \tag{4}$$

Please note that this changing of the summation counter leads immediately to the conclusion that there is no unique way of writing down the final answer in summation form. However, there is usually a best version of the solution.

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2. Following on from the previous question, differentiate the Binomial Series form of $(1+x)^{-3}$ to find the series for $(1+x)^{-4}$. Confirm this by using the general expression for the Binomial Series.

A2. If we use the following,

$$(1+x)^{-3} = 1 - 3x + \frac{3\cdot 4}{2}x^2 - \frac{4\cdot 5}{2}x^3 + \frac{5\cdot 6}{2}x^4 - \frac{6\cdot 7}{2}x^5 + \cdots,$$

as the series, then its derivative is,

$$-3(1+x)^{-4} = -3 + \frac{2\cdot3\cdot4}{2}x - \frac{3\cdot4\cdot5}{2}x^2 + \frac{4\cdot5\cdot6}{2}x^3 - \frac{5\cdot6\cdot7}{2}x^4 + \cdots,$$

which, when rearranged, gives

$$(1+x)^{-4} = 1 - \frac{2\cdot3\cdot4}{3!}x + \frac{3\cdot4\cdot5}{3!}x^2 - \frac{4\cdot5\cdot6}{3!}x^3 + \frac{5\cdot6\cdot7}{3!}x^4 + \cdots.$$

This could be rewritten as

$$(1+x)^{-4} = \frac{1}{3!}\sum_{n=0}^{\infty}(-1)^n(n+1)(n+2)(n+3)x^n,$$

or even as

$$(1+x)^{-4} = \frac{1}{3!}\sum_{n=0}^{\infty}(-1)^n\frac{(n+3)!}{n!}x^n.$$

To check, we can use the Binomial Series formula with $n=-4$; we have,

$$\begin{aligned}(1+x)^{-4} &= 1 + (-4)x + \frac{(-4)(-5)}{2}x^2 + \frac{(-4)(-5)(-6)}{2\cdot3}x^3 + \frac{(-4)(-5)(-6)(-7)}{2\cdot3\cdot4}x^4 \\ &\quad + \frac{(-4)(-5)(-6)(-7)(-8)}{2\cdot3\cdot4\cdot5}x^5 + \cdots \\ &= 1 - 4x + \frac{4\cdot5}{2}x^2 - \frac{4\cdot5\cdot6}{2\cdot3}x^3 + \frac{5\cdot6\cdot7}{2\cdot3}x^4 - \frac{6\cdot7\cdot8}{2\cdot3}x^5 + \cdots \\ &= \frac{3!}{0!\,3!} - \frac{4!}{1!\,3!}x + \frac{5!}{2!\,3!}x^2 - \frac{6!}{3!\,3!}x^3 + \frac{7!}{4!\,3!}x^4 + \cdots \\ &= \frac{1}{3!}\sum_{n=0}^{\infty}\frac{(-1)^n(n+3)!}{n!}x^n.\end{aligned}$$

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3. Find the binomial expansions for

(a) $(1+x)^{-1/2}$, **(b)** $(1+x)^{1/2}$ and **(c)** $(1+x)^{-3/2}$.

Note that part (a) is in the lecture notes, but do resist the temptation to look before trying this out! Check your answers to (b) and (c) by the respective integration and differentiation of the result of (a).

A3. **(a)** Here we have $n = -1/2$ and therefore the expansion is,

$$\begin{aligned}(1+x)^{-1/2} &= 1 + (-\tfrac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}x^4 + \cdots \\ &= 1 - \tfrac{1}{2}x + \frac{1.3}{2^2\,2!}x^2 - \frac{1.3.5}{2^3\,3!}x^3 + \frac{1.3.5.7}{2^4\,4!}x^4 + \cdots\end{aligned}$$

Therefore the general term may be written as

$$
\begin{aligned}
& (-1)^n \frac{1.3.5.\cdots.(2n-3).(2n-1)}{2^n\, n!} x^n \\
= \;& (-1)^n \frac{1.2.3.4.5.6.\cdots.(2n-2).(2n-1).(2n)}{2^n\, n!\,[2.4.6.\cdots.(2n-2).(2n)]} x^n \\
= \;& (-1)^n \frac{(2n)!}{2^n\, n!\,(2^n\, n!)} x^n \\
= \;& (-1)^n \frac{(2n)!}{(2^n\, n!)^2} x^n
\end{aligned}
$$

Therefore the power series representation is,

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}\, n!\, n!} x^n.$$

We have taken a very slightly different route from the lectures where we considered the 4^{th} term in the series, tidied up the coefficient and then used that to determine an expression for the general coefficient.

(b) In this case we use $n = 1/2$, and given that this is also an odd multiple of $1/2$, we should anticipate an almost identical analysis to that for part (a). So we get,

$$(1+x)^{1/2} = 1+(\tfrac{1}{2})x+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{5!}x^5+\cdots. \quad (1)$$

Concentating on the coefficient of x^4 we have,

$$\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!} = \tfrac{1}{2}(-1)^3\frac{1\cdot 3\cdot 5}{2^3 4!} = \tfrac{1}{2}(-1)^3\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6\; 2^3 4!} = \tfrac{1}{2}(-1)^3\frac{6!}{2^6 3!\, 4!}$$

Therefore we can see that the coefficient of x^n is

$$\tfrac{1}{2}(-1)^{n+1}\frac{(2n-2)!}{2^{2n-2}(n-1)!\, n!}.$$

It is important to note that the constant term in (1) has the same sign as the x-term, and therefore we will have to leave this out of the summation. Hence we have,

$$(1+x)^{1/2} = 1+\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-2)!}{2^{2n-2}(n-1)!\, n!}x^n.$$

(c) The solution for this one is that,

$$(1+x)^{-3/2} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n+2)!}{2^{2n+1}\, n!\,(n+1)!}x^n.$$

For comparison, the other two are,

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}\, n!\, n!}x^n$$

and

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{2^{2n-1}(n-1)!\, n!} x^n.$$

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4. Express $(1+x)^{-1/3}$ as a Binomial Series. Note that this one cannot be expressed in summation form....unless you decide to define your own notation! [Hint: do a google search for what are called double factorials.]

A4. The solution is

$$(1+x)^{-1/3} = 1 - \left(\frac{x}{3}\right) + \frac{1.4}{2!}\left(\frac{x}{3}\right)^2 - \frac{1.4.7}{3!}\left(\frac{x}{3}\right)^3 + \frac{1.4.7.10}{4!}\left(\frac{x}{3}\right)^4 + \cdots, \qquad (1)$$

and this came directly from the Binomial Series with $n = -1/3$.

We can't manipulate these coefficients using factorials, as we did for some of the previous series. However, the *hint* was to check out double factorials. These turn out to be like standard factorials, but they miss out every other number. So we have,

$$9!! = 9 \cdots 7 \cdots 5 \cdots 3 \cdots 1 \qquad \text{and} \qquad 10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2.$$

This means that the answer to Q3a which started off as,

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{1.3}{2^2\, 2!}x^2 - \frac{1.3.5}{2^3\, 3!}x^3 + \frac{1.3.5.7}{2^4\, 4!}x^4 + \cdots, \qquad (2)$$

could be written in the alternative form,

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{3!!}{2^2\, 2!}x^2 - \frac{5!!}{2^3\, 3!}x^3 + \frac{7!!}{2^4\, 4!}x^4 + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n\, n!} x^n, \qquad (3)$$

rather in than the final form that we found,

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}\, n!\, n!} x^n. \qquad (4)$$

The question is now: is (3) better than (4)? Both are correct, but people tend not to know about double factorials.

But for the present problem we have products of integers which form a sequence with a difference of 3. Given that double factorials are apparently well-known (!!) the question is: are there triple factorials? It turns out that there is. Here are three examples:

$$10!!! = 10 \cdot 7 \cdot 4 \cdot 1, \qquad 11!!! = 11 \cdot 8 \cdot 5 \cdot 2 \qquad \text{and} \qquad 12!!! = 12 \cdot 9 \cdot 6 \cdot 3.$$

Hence the solution which we seek and which is given in Eq. (1), may be written in the form,

$$(1+x)^{-1/3} = \sum_{n=0}^{\infty} (-1)^n \frac{(3n-2)!!!}{n!} \left(\frac{x}{3}\right)^n.$$

For this one, there is no alternative compact form.

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5. Find the Binomial Series representations for **(a)** $(1+2x)^{-1}$ and **(b)** $(3+2x)^{-1}$.

A5. We begin by requoting the formula for the Binomial Series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots .$$

(a) $y = (1+2x)^{-1}$. So we let $n=-1$ in the Binomial Series and replace x by $2x$. Hence,

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + \cdots .$$

Hence

$$(1+2x)^{-1} = 1 - 2x + (2x)^2 - (2x)^3 + (2x)^4 + \cdots = \sum_{n=0}^{\infty}(-2x)^n.$$

(b) Here we have $y=(3+2x)^{-1}$. The leading 3 is the problem here because the Binomial Series is defined with a leading 1. Therefore we must also have a leading 1. So we can rewrite this as,

$$y = \tfrac{1}{3}(1+\tfrac{2}{3}x)^{-1}$$

Hence,

$$(3+2x)^{-1} = \tfrac{1}{3}\Big[1 + (-1)(\tfrac{2}{3}x) + \frac{(-1)(-2)}{2!}(\tfrac{2}{3}x)^2 + \frac{(-1)(-2)(-3)}{3!}(\tfrac{2}{3}x)^3 + \cdots\Big],$$

or

$$(3+2x)^{-1} = \tfrac{1}{3}\Big[1 - \tfrac{2}{3}x + (\tfrac{2}{3}x)^2 - (\tfrac{2}{3}x)^3 + (\tfrac{2}{3}x)^4 + \cdots\Big] = \tfrac{1}{3}\sum_{n=0}^{\infty}(-\tfrac{2}{3}x)^n.$$

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6. Obtain the binomial series for $(1+x^2)^{-1}$ and hence find a power series representation for $\tan^{-1}x$. Use this result to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

A6. If $y(x) = \tan^{-1}x$, then

$$y'(x) = (1+x^2)^{-1},$$

using the methods from the Differentiation part of the unit — this is why we need a binomial series for $(1+x^2)^{-1}$.

We have already seen that $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$, and therefore we can replace all instances of x by x^2 to obtain,

$$y'(x) = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots .$$

Now we can integrate term-by-term:

$$y(x) = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots .$$

Note that the constant of integration has already been set to zero since $\tan^{-1}0 = 0$ — this constant **MUST** be calculated even if it turns out to be zero. The required result for $\frac{\pi}{4}$ is obtained on setting $x=1$:

$$\tan^{-1}1 = \tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots .$$

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7. **(a)** Use the chain rule (i.e. implicit differentiation) to find the derivative of $y = \tanh^{-1} x$. Use this result and the Binomial Series to find a power series representation for y.

(b) At some point in part (a) you will have an expression for x as an explicit function of y; replace the sinh and cosh terms by the correct exponentials, solve for e^y and hence find a different expression for y from the one given initially.

(c) Use the results of parts (a) and (b) to show that,

$$\tfrac{1}{2} + \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} + \frac{(\frac{1}{2})^7}{7} + \cdots = \tfrac{1}{2}\ln 3.$$

A7. This question has some similarities to the one involving $\tan^{-1} x$.

(a) If $y = \tanh^{-1} x$, then $\tanh y = x$. Given that

$$\tanh y = \frac{\sinh y}{\cosh y},$$

then,

$$\frac{d(\tanh y)}{dy} = \frac{d}{dy}\Big(\frac{\sinh y}{\cosh y}\Big) = \frac{\cosh^2 y - \sinh^2 y}{\cosh^2 y} = 1 - \tanh^2 y.$$

Now we may differentiate $\tanh y = x$ with respect to x (chain rule):

$$\frac{d}{dx}\tanh y = \frac{dy}{dx}\frac{d(\tanh y)}{dy} = (1 - \tanh^2 y)\frac{dy}{dx} = 1.$$

Hence

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2} = (1 - x^2)^{-1}.$$

Now we are in a position to use the Binomial Series. We know that,

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots ,$$

and therefore

$$\frac{dy}{dx} = (1 - x^2)^{-1} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots .$$

Integrating once and using the fact that $y(0) = 0$ to find the constant of integration, we obtain,

$$y = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots .$$

This is the power series representation for $\tanh^{-1} x$. Do compare this with the earlier power series for $\tan^{-1} x$.

(b) We have

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

Rearrangement of this expression to give e^{2y} in terms of x yields,

$$e^{2y} = \frac{1 + x}{1 - x}.$$

Hence

$$y = \tfrac{1}{2} \ln \frac{1+x}{1-x}.$$

Note that $\tanh y$ lies between -1 and $+1$ only, and hence $-1 < x < 1$. This ensures that $(1+x)/(1-x)$ is positive in that range of values of x.

(c) Parts (a) and (b) tell us that,

$$\tfrac{1}{2} \ln \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots .$$

If we set $x = \frac{1}{2}$ in this equation then,

$$\tfrac{1}{2} + \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} + \frac{(\frac{1}{2})^7}{7} + \frac{(\frac{1}{2})^9}{9} + \cdots = \tfrac{1}{2} \ln 3.$$

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8. **[For fun only!]** Use the binomial expansion of $(1+x)^n$, where n is a positive integer, to show that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \qquad \text{and} \qquad \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

A8. Using the Binomial Theorem we have

$$(1+x)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} + \cdots + \binom{n}{n-1} x + \binom{n}{n},$$

and therefore when $x = 1$ we have the first result immediately, and the second follows when $x = -1$.

We may check this using a random row in Pascal's triangle. Take the 5th and 6th rows as typical examples:

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1,$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

Then,

$$1+5+10+10+5+1 = 32 = 2^5$$

$$1+6+15+20+15+6+1 = 64 = 2^6$$

and

$$1-5+10-10+5-1 = 0$$

$$1-6+15-20+15-6+1 = 0.$$

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9. **[For fun only!]** Write out the Binomial Expansions for both $(1+x)^n$ and $(1+x^{-1})^n$ where n is an integer. Use these results to show that

$$\sum_{r=0}^{n}\left[\binom{n}{r}\right]^2 = \binom{2n}{n}.$$

[Hint: where might the Binomial coefficient on the right hand side have come from?]

A9. We have

$$(1+x)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}x + \binom{n}{n} \tag{1}$$

and

$$(1+x^{-1})^n = \binom{n}{0}x^{-n} + \binom{n}{1}x^{-n+1} + \cdots + \binom{n}{n-1}x^{-1} + \binom{n}{n}. \tag{2}$$

Clearly, the product of the two expressions on the left hand sides of (1) and (2) must be equal to the products of the right hand sides.

If we multiply the right hand sides of (1) and (2), the result will be a polynomial where the terms range from x^{-n} to x^n. It would be almost ridiculous to write all of these down, but the summation that we are seeking corresponds to the coefficients of x^0.

Therefore we need to find the x^0 coefficient of the product of the left hand sides of (1) and (2):

$$(1+x)^n(1+x^{-1})^n.$$

Now we may tranform this into,

$$\frac{(1+x)^{2n}}{x^n}. \tag{3}$$

The term that we want is the coefficient of x^n in the Binomial expansion of $(1+x)^{2n}$ which is

$$\binom{2n}{n}.$$

Hence

$$\sum_{r=0}^{n}\left[\binom{n}{r}\right]^2 = \binom{2n}{n}.$$

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10. **[For fun only!]** Try to determine a general formula for the coefficient of $x^i y^j z^k$ in the expansion of $(x+y+z)^n$, where $i+j+k=n$. This expansion is called a Trinomial expansion.

A10. We begin by expanding $(x+y+z)^n$ as a binomial in x and $(y+z)$:

$$
\begin{aligned}
(x+y+z)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}(y+z)^1 + \binom{n}{2}x^{n-2}(y+z)^2 + \binom{n}{3}x^{n-3}(y+z)^3 + \cdots \\
&= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}\Big[\binom{1}{0}y + \binom{1}{1}z\Big] \\
&\qquad + \binom{n}{2}x^{n-2}\Big[\binom{2}{0}y^2 + \binom{2}{1}yz + \binom{2}{2}z^2\Big] \\
&\qquad + \binom{n}{3}x^{n-3}\Big[\binom{3}{0}y^3 + \binom{3}{1}y^2z + \binom{3}{2}yz^2 + \binom{3}{3}z^3\Big] + \cdots
\end{aligned}
$$

The form of the coefficient of $x^{n-3}y^2z$, which is $\binom{n}{3}\binom{3}{1}$, suggests the general term,

$$\binom{n}{m}\binom{m}{k}x^{n-m}y^{m-k}z^k. \qquad (*)$$

Now we shall attempt to force the above expression to be equivalent to a formula involving $x^iy^jz^k$, as requested at the start of the question. Therefore we shall let,

$$i = n-m \qquad \text{and} \qquad j = m-k.$$

These are equivalent to,

$$n = i+j+k \qquad \text{and} \qquad m = j+k.$$

Hence expression $(*)$ becomes

$$\binom{i+j+k}{j+k}\binom{j+k}{k}x^iy^jz^k.$$

Finally, we need to expand the product of the Binomial coefficients:

$$
\begin{aligned}
\binom{i+j+k}{j+k}\binom{j+k}{k} &= \frac{(i+j+k)!}{i!\,(j+k)!} \times \frac{(j+k)!}{j!\,k!} \\
&= \frac{(i+j+k)!}{i!\,j!\,k!} \\
&= \frac{n!}{i!\,j!\,k!}.
\end{aligned}
$$

Therefore the coefficient of $x^iy^jz^k$ is $\dfrac{n!}{i!\,j!\,k!}$.

One could, should one wish, write the following:

$$(x+y+z)^n = \underbrace{\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}}_{i+j+k=n}\frac{n!}{i!\,j!\,k!}x^iy^jz^k.$$

I would say that this is probably the most difficult/awkward problem that I have set you so far, at least conceptually.