Department of Mechanical Engineering, University of Bath

Engineering Mathematics S1 ME12002

Problem sheet 7 — Series – Taylor's series, convergence, l'Hôpital's rule

1. Find a cubic polynomial, $f(x)$, which is such that

(a) $f(1) = 2$, $f'(1) = -1$, $f''(1) = 0$ and $f'''(1) = 6$;

(b) $f(0) = 3$, $f'(0) = 0$, $f''(0) = -6$ and $f'''(0) = 12$;

(c) $f(2) = 7$, $f'(2) = 12$, $f''(2) = 18$ and $f'''(2) = 12$.

In parts (a) and (c) rewrite the polynomials as finite power series in x.

A1. For this question, we merely write out the definition of the Taylor's series as far as the cubic term, and substitute the given data.

$$f(x) = f(a) + f'(a)\,(x-a) + \tfrac{1}{2}f''(a)\,(x-a)^2 + \tfrac{1}{6}f'''(a)\,(x-a)^3. \qquad (1)$$

(a On filling in the values of f and its derivatives at $x = 1$ and with $a = 1$, we get,

$$\begin{aligned} f(x) &= 2 - (x-1) + (x-1)^3 \\ &= 2 + 2x - 3x^2 + x^3. \end{aligned}$$

(b) For this one we have $a = 0$, and hence,

$$f(x) = 3 - 3x^2 + 2x^3.$$

(c) For this one we have $a = 2$, and hence,

$$\begin{aligned} f(x) &= 7 + 12(x-2) + 9(x-2)^2 + 2(x-1)^3 \\ &= 3 - 3x^2 + 2x^3. \end{aligned}$$

Curiously, the answers to parts (b) and (c) are identical. Perhaps that might have been planned!

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2. **(A strange one!)** Find a way to use Taylor's series to rewrite $y = 1 + x + x^2 + x^3$ as a power series in $(x-1)$. By this I mean that $y = \sum_{n=0}^{3} a_n (x-1)^n$.

A2. We have $y = 1 + x + x^2 + x^3$. This may be written as a sum of powers of $(x-1)$ by first evaluating y and its derivatives at $x = 1$. So

$$\begin{aligned} y = 1 + x + x^2 + x^3 \quad &\Rightarrow \quad y(1) = 4 \\ y' = 1 + 2x + 3x^2 \quad &\Rightarrow \quad y'(1) = 6 \\ y'' = 2 + 6x \quad &\Rightarrow \quad y''(1) = 8 \\ y''' = 6 \quad &\Rightarrow \quad y'''(1) = 6 \end{aligned}$$

Hence we have,

$$y = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3.$$

Hmm, there are some binomial coefficients lurking in the answer....

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3. Find the Taylor's series of $y = (1+x)^{-2}$ about $x = 0$ and the one about $x = 1$. What are their radii of convergence?

A3. We shall create a Table of data to substitute into the definition of the Taylor's series.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(1)$
0	$+(1+x)^{-2}$	$+1$	$+1/2^2$
1	$-2(1+x)^{-3}$	-2	$-2/2^3$
2	$+3!(1+x)^{-4}$	$+3!$	$+3!/2^4$
3	$-4!(1+x)^{-5}$	$-4!$	$-4!/2^5$
4	$+5!(1+x)^{-6}$	$+5!$	$+5!/2^6$

Note that I haven't simpified any of the entries in this Table by multiplying them out and/or performing the quotients. Thus the Table retains inormation about the general pattern of the coefficients.

For the Taylor's series about $x = 0$ (or the Maclaurin expansion) we have,

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

The general term is $u_n = (-1)^n (n+1) x^n$ and so $u_{n+1} = (-1)^{n+1}(n+2)x^{n+1}$. Hence

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|-\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \left|\frac{(1+2/n)x}{(1+1/n)}\right| \longrightarrow |x| \text{ as } n \to \infty.$$

Hence the radius of convergence is 1.

For the Taylor's series about $x = 1$ we have,

$$(1+x)^{-2} = \frac{1}{4}\Big[1 - \frac{2}{2}(x-1) + \frac{3}{2^2}(x-1)^2 - \frac{4}{2^3}(x-1)^3 + \frac{5}{2^4}(x-1)^4 + \cdots\Big] = \tfrac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\Big(\frac{x-1}{2}\Big)^n.$$

So the general term is $u_n = (-1)^n (n+1)(x-1)^n / 2^{n+2}$. Therefore,

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|-\frac{(n+2)2^n (x-1)^{n+1}}{(n+1)2^{n+1}(x-1)^n}\right| \longrightarrow \left|\frac{x-1}{2}\right| \text{ as } n \to \infty.$$

Hence the radius of convergence is 2. More specifically we have $|x-1| < 2$ for convergence or, if x is confined to be real, $-1 < x < 3$.

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4. Find the Taylor's series of $(1+x)^{-1}$ about $x=2$. Find its radius of convergence. It is possible to use the Binomial series to find this power series, but it may need a little bit of thinking...

A4. We shall again create a Table of data to substitute into the definition of the Taylor's series.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$+(1+x)^{-1}$	$+1/3$
1	$-(1+x)^{-2}$	$-1/3^2$
2	$+2(1+x)^{-3}$	$+2!/3^3$
3	$-3!(1+x)^{-4}$	$-3!/3^4$
4	$+4!(1+x)^{-5}$	$+4!/3^5$

Hence the Taylor's series about $x=2$ is

$$(1+x)^{-1} = \frac{1}{3}\Big[1-\frac{1}{3}(x-2)+\frac{1}{3^2}(x-2)^2-\frac{1}{3^3}(x-2)^3+\frac{1}{3^4}(x-2)^4+\cdots\Big] = \tfrac{1}{3}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{x-2}{3}\Big)^n.$$

So the general term is $u_n = (-1)^n(x-2)^n/3^{n+1}$. Therefore,

$$\Big|\frac{u_{n+1}}{u_n}\Big| = \Big|-\frac{3^{n+1}(x-2)^{n+1}}{3^{n+2}(x-2)^n}\Big| \longrightarrow \Big|\frac{x-2}{3}\Big| \text{ as } n\to\infty.$$

Hence the radius of convergence is 3. More specifically we have $|x-2|<3$ for convergence. When x is real this means that $-1<x<5$ for convergence.

With respect to how to use the Binomial series to solve this problem, I will begin by quoting the following, which has appeared a few times already:

$$(1+y)^{-1} = 1-y+y^2-y^3+y^4+\cdots\,. \tag{1}$$

Now we may manipulate $(1+x)^{-1}$ into a suitable form, as follows:

$$(1+x)^{-1} = \Big(3+(x-2)\Big)^{-1} = \tfrac{1}{3}\Big(1+\frac{(x-2)}{3}\Big)^{-1}.$$

We may now let $y=(x-2)/3$ in Eq. (1) and hence

$$(1+x)^{-1} = \frac{1}{3}\Big[1-\Big(\frac{x-2}{3}\Big)^1+\Big(\frac{x-2}{3}\Big)^2-\Big(\frac{x-2}{3}\Big)^3+\Big(\frac{x-2}{3}\Big)^4+\cdots\Big].$$

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5. Find the Maclaurin series for $\sin ax$ and use this to find the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(ax)}{x}$, (b) $\displaystyle\lim_{x\to 0}\frac{\sin(ax)-ax}{x^3}$.

A5. The Maclaurin's series for $\sin ax$ appears in the lecture notes, so I'll just quote it:

$$\sin ax = ax - \frac{a^3x^3}{3!}+\frac{a^5x^5}{5!}-\frac{a^7x^7}{7!}+\cdots\,.$$

(a) So

$$\lim_{x\to 0}\frac{\sin ax}{x} = \lim_{x\to 0}\left[\frac{ax-\frac{a^3x^3}{3!}+\frac{a^5x^5}{5!}-\frac{a^7x^7}{7!}+\cdots}{x}\right]$$

$$= \lim_{x\to 0}\left[a-\frac{a^3x^2}{3!}+\frac{a^5x^4}{5!}+\cdots\right]$$

$$= a.$$

(b) Also

$$\lim_{x\to 0}\frac{\sin ax-ax}{x^3} = \lim_{x\to 0}\left[\frac{\cancel{ax}-\frac{a^3x^3}{3!}+\frac{a^5x^5}{5!}-\frac{a^7x^7}{7!}+\cdots\cancel{-ax}}{x^3}\right]$$

$$= \lim_{x\to 0}\left[-\frac{a^3}{3!}+\frac{a^5x^2}{5!}+\cdots\right]$$

$$= -a^3/6.$$

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6. Find the Maclaurin's series for both (a) $\cos(ax)$ and (b) $\cosh(ax)$. Use these results (not l'Hôpital's rule) to determine,

$$\lim_{x\to 0}\frac{\cos(ax)+\cosh(ax)-2}{x^4}.$$

You may, of course, use l'Hôpital's rule to confirm your solutions.

A6. Sorry, but I shall quote these two Maclaurin series:

$$\cos(ax) = 1-\frac{(ax)^2}{2!}+\frac{(ax)^4}{4!}-\frac{(ax)^6}{6!}+\cdots,$$

$$\cosh(ax) = 1+\frac{(ax)^2}{2!}+\frac{(ax)^4}{4!}+\frac{(ax)^6}{6!}+\cdots.$$

The first one was derived in the lectures. The second may be derived in almost exactly the same way, but do note the different patterns of signs between these two functions.

This means that,

$$\cos(ax)+\cosh(ax) = 2\left[1+\frac{(ax)^4}{4!}+\frac{(ax)^8}{8!}+\cdots\right],$$

and hence,

$$\cos(ax)+\cosh(ax)-2 = 2\left[\frac{(ax)^4}{4!}+\frac{(ax)^8}{8!}+\cdots\right],$$

and

$$\frac{\cos(ax)+\cosh(ax)-2}{x^4} = 2\left[\frac{a^4}{4!}+\frac{a^8x^4}{8!}+\cdots\right].$$

Finally,

$$\lim_{x\to 0}\frac{\cos(ax)+\cosh(ax)-2}{x^4} = \tfrac{1}{12}a^4.$$

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7. Find the Maclaurin series representation of $\ln(2+x)$. What is its radius of convergence?

A7. We shall create a Table of data to substitute into the definition of the Taylor's series.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(2+x)$	$\ln 2$
1	$+(2+x)^{-1}$	$+2^{-1}$
2	$-(2+x)^{-2}$	-2^{-2}
3	$+2(2+x)^{-3}$	$+2\times 2^{-3}$
4	$-3!(2+x)^{-4}$	$-3!\times 2^{-4}$
5	$+4!(2+x)^{-5}$	$+4!\times 2^{-5}$

From this we see that $f^{(n)}(0)=(-1)^{n+1}(n-1)!\,2^{-n}$. Hence the series is,

$$\ln(2+x)=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!\,2^{-n}x^n}{n!},$$

or

$$\ln(2+x)=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n\,2^n}.$$

In the above, note that $(n-1)!/n! = 1/n$. Note also that the very first term doesn't follow the general pattern and therefore it is not included within the summation.

Given that $\left|\dfrac{(-1)^{n+2}x^{n+1}n\,2^n}{(-1)^{n+1}x^n(n+1)2^{n+1}}\right| \longrightarrow \dfrac{|x|}{2}$ as $n\longrightarrow\infty$, the radius of convergence is 2.

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8. In the lecture notes we found a power series representation of e^{-x^2} by first finding a Maclaurin expansion of e^{-x} followed by the replacing of x by x^2. It was also mentioned that the direct determination of the Maclaurin series of e^{-x^2} is much more challenging. So here's your chance! Use the definition of the Maclaurin series to determine the power series of e^{-x^2} up to and including the x^6 term. If you have the stamina, go as far as the x^8 term!

A8. O.k. here goes. I will list the function and its first 14 derivatives.

$$
\begin{aligned}
y &= e^{-x^2} \\
y' &= e^{-x^2}\left[-2x\right] \\
y'' &= e^{-x^2}\left[-2+4x^2\right] \\
y''' &= e^{-x^2}\left[12x-8x^3\right] \\
y^{(4)} &= e^{-x^2}\left[12-48x^2+16x^4\right] \\
y^{(5)} &= e^{-x^2}\left[-120x+160x^3-32x^5\right] \\
y^{(6)} &= e^{-x^2}\left[-120+720x^2-480x^4+64x^6\right] \\
y^{(7)} &= e^{-x^2}\left[1680x-3360x^3+1344x^5-128x^7\right] \\
y^{(8)} &= e^{-x^2}\left[1680-13440x^2+13440x^4-3584x^6+256x^8\right] \\
y^{(9)} &= e^{-x^2}\left[-30240x+80640x^3-48384x^5+9216x^7-512x^9\right] \\
y^{(10)} &= e^{-x^2}\left[-30240+302400x^2-403200x^4-161280x^6-23040x^8-1024x^{10}\right] \\
y^{(11)} &= e^{-x^2}\left[665280x-2217600x^3+1774080x^5-506880x^7+56320x^9-2048x^{11}\right] \\
y^{(12)} &= e^{-x^2}\left[665280-7983360x^2+13305600x^4-7096320x^6+1520640x^8-135168x^{10}+4096x^{12}\right]
\end{aligned}
$$

Hence the odd-numbered derivatives are all zero at $x=0$, and we have

$$
y(0)=1, \qquad y''(0)=-2, \qquad y^{(4)}(0)=12, \qquad y^{(6)}(0)=-120, \qquad y^{(8)}(0)=1680,
$$
$$
y^{(10)}(0)=-30240, \qquad y^{(12)}=665280.
$$

When substituted into the formula for the Maclaurin series these give,

$$
y = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \cdots,
$$

and therefore I would conclude that this verifies that the shortcut we took yields the correct solution. Apart from the length of this analysis and the huge number of places where the arithmetic could go wrong, there isn't an immediately obvious way to show that the formula that we have found does indeed have these factorials in the denominator.

.....and I have to come clean with you all. While I did calculate the above by hand up to the $y^{(8)}$ term, I decided to write a computer program both to check my arithmetic and to extend the data. The values of $y^{(n)}$ at $x=0$ are, in fact, $n!/(\frac{1}{2}n)!$ when n is even and zero otherwise. The summation version of the final solution is,

$$
y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!},
$$

although this shouldn't be a surprise given the analysis in the lecture.

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9. Determine the convergence properties of the following numerical series.

(a) $\dfrac{2}{1^3} + \dfrac{3}{2^3} + \dfrac{4}{3^3} + \dfrac{5}{4^3} + \cdots$ (b) $\dfrac{1}{10} - \dfrac{2}{11} + \dfrac{3}{12} - \dfrac{4}{13} + \cdots$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^p}{k!} \quad (p>0)$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(2k)!}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{k!\,k!}{(2k)!}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

In part (f) you will need to use the result that $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, which is to be proven in a later question.

A9. Six cases to consider.

(a) The general term is $u_n = \dfrac{n+1}{n^3}$. D'Alembert's test gives

$$\begin{aligned}\left|\frac{u_{n+1}}{u_n}\right| &= \frac{(n+2)}{(n+1)^3} \times \frac{n^3}{(n+1)} \\ &= \frac{n^4+2n^3}{(n+1)^4} \\ &= \frac{1+{}^2\!/\!_n}{(1+{}^1\!/\!_n)^4}\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1.$$

This means that the test isn't conclusive. In practice this one does converge, but d'Alembert's test is no good for decision-making purposes.

(b) In this case both the numerator and denominator increase by 1 for each succeeding term, and therefore the terms tend upwards towards 1:

$$|u_n| = \frac{n}{n+9} = \frac{1}{1+{}^9\!/\!_n} \to 1 \quad \text{as } n \to \infty,$$

which suggests divergence. The ratio for d'Alembert's test also tends towards 1, which is formally inconclusive.

(c) We have $u_k = \dfrac{k^p}{k!}$. Therefore,

$$\begin{aligned}\left|\frac{u_{k+1}}{u_k}\right| &= \frac{(k+1)^p\ k!}{(k+1)!\ k^p} \\ &= \left(\frac{k+1}{k}\right)^p \frac{1}{(k+1)} \\ &= \left(1+\frac{1}{k}\right)^p \frac{1}{(k+1)} \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Therefore this series converges.

(d) We have $u_k = \dfrac{k!}{(2k)!}$. Therefore,

$$
\begin{aligned}
\left|\frac{u_{k+1}}{u_k}\right| &= \frac{(k+1)!\ (2k)!}{(2k+2)!\ k!} \\
&= \frac{k+1}{(2k+2)(2k+1)} \\
&= \frac{1}{2(2k+1)} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}
$$

Therefore this series also converges.

(e) We have $u_k = \dfrac{k!\,k!}{(2k)!}$. Therefore,

$$
\begin{aligned}
\left|\frac{u_{k+1}}{u_k}\right| &= \frac{(k+1)!\,(k+1)!\ (2k)!}{(2k+2)!\ k!\,k!} \\
&= \frac{(k+1)^2}{(2k+2)(2k+1)} \\
&= \frac{k+1}{2(2k+1)} \\
&\to {}^1\!/\!_4 \quad \text{as } k \to \infty.
\end{aligned}
$$

Therefore this series also converges.

(f) We have $u_n = \dfrac{n!}{n^n}$. Therefore,

$$
\begin{aligned}
\left|\frac{u_{n+1}}{u_n}\right| &= \frac{(n+1)!\,n^n}{(n+1)^{n+1}\,n!} \\
&= \frac{(n+1)n^n}{(n+1)^{n+1}} \\
&= \frac{n^n}{(n+1)^n} \\
&= \frac{1}{(1+{}^1\!/\!_n)^n}.
\end{aligned}
$$

Now we have to use the result given in the question, namely, that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Therefore we get,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{e}.$$

This is less than 1, and so the series converges.

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10. Determine the radii of convergence of the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (c) $\displaystyle\sum_{n=1}^{\infty} n^{100}x^n$

(d) $\cosh\sqrt{x} = 1 + \dfrac{x}{2!} + \dfrac{x^2}{4!} + \dfrac{x^3}{6!} + \cdots$ (e) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\cdots$ (f) $\displaystyle\sum_{n=1}^{\infty}(-1)^n 2^n x^{2n}$

(g) $\displaystyle\sum_{n=1}^{\infty} n!\, x^n$ (h) $\displaystyle\sum_{n=1}^{\infty}\frac{n!\, x^n}{n^n}$ (i) $\displaystyle\sum_{n=0}^{\infty}\frac{(1+2n)}{(1+2^n)}x^{n/3}$

A10. We apply the standard d'Alembert convergence test.

(a) We have $u_n = x^n$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|.$$

Therefore we need $|x| < 1$ for convergence and hence the radius of convergence is 1.

(b) We have $u_n = nx^n$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = |(1+{}^1\!/\!_n)x| \to |x| \quad \text{as } n\to\infty.$$

Therefore the radius of convergence is also 1.

(c) We have $u_n = n^{100}x^n$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)^{100}x^{n+1}}{n^{100}x^n}\right| = |(1+{}^1\!/\!_n)^{100}x| \to |x| \quad \text{as } n\to\infty.$$

Once more, the radius of convergence is 1. This result indicates that $u_n = n^\alpha x^n$ will have a unit radius of convergence irrespective of how large α is.

(d) We have $u_n = \dfrac{x^n}{(2n)!}$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}\,(2n)!}{(2n+2)!\,x^n}\right| = \left|\frac{x}{(2n+2)(2n+1)}\right| \to 0 \quad \text{as } n\to\infty.$$

It is clear that d'Alembert's criterion is satisfied independently of the value of x. Therefore this series has an infinite radius of convergence.

(e) The Taylor's series may be written in the form,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Therefore $u_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$ and so

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \left|\frac{x^{2n+2}(2n)!}{x^{2n}(2n+2)!}\right| \quad = \quad \left|\frac{x^2}{(2n+2)(2n+1)}\right| \quad \to \quad 0 \quad \text{as } n\to\infty.$$

Therefore this series also has an infinite radius of convergence.

(f) We have $u_n = (-1)^n 2^n x^{2n}$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \left|\frac{2^{n+1}x^{2n+2}}{2^n x^{2n}}\right| \quad = \quad |2x^2|.$$

Therefore we need $2|x|^2 < 1$ for convergence, which implies that $|x| < 2^{-1/2}$.

(g) We have $u_n = n!\,x^n$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| \quad = \quad |(n+1)x| \quad \to \quad \infty \quad \text{as } n\to\infty.$$

This series fails d'Alembert's test for all nonzero values of x and therefore its radius of convergence is zero. This means that the series never converges, although the trivial case, $x=0$, is the sole value where convergence happens.

(h) We have $u_n = \dfrac{n!\,x^n}{n^n}$, which gives

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \left|\frac{(n+1)!\,n^n x^{n+1}}{n!\,(n+1)^{n+1}x^n}\right| \quad = \quad \left|\frac{n^n x}{(n+1)^n}\right| \quad = \quad |(1+1/n)^{-n}x| \quad \to \quad |x/e| \quad \text{as } n\to\infty.$$

Given that convergence corresponds to $|x/e| < 1$, the radius of convergence is e.

(i) We have the general term, u_n, contained in the summation. Hence d'Alembert's ratio is,

$$\left|\frac{(3+2n)x^{(n+1)/3}(1+2^n)}{(1+2n)x^{n/3}(1+2^{n+1})}\right|.$$

We shall process this by multiplying both the numerator and denominator by, $x^{-n/3}2^{-n}/n$. Therefore d'Alembert's ratio is now,

$$\left|\frac{(2+3/n)x^{1/3}(1+2^{-n})}{(2+1/n)(2+2^{-n})}\right|.$$

As $n\to\infty$ this quotient tends towards, $|x^{1/3}|/2$, which must be less than 1 for convergence. Hence the radius of convergence is 8.

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11. Use l'Hôpital's rule to find the following limits.

(a) $\displaystyle\lim_{x\to\frac{1}{2}} \frac{\cos \pi x}{2x-1}$ (b) $\displaystyle\lim_{x\to 1} \frac{\sin \pi x}{\sqrt{x^2-1}}$ (c) $\displaystyle\lim_{x\to 0} \frac{\cosh ax - 1}{x}$

(d) $\displaystyle\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+2x+1}$ (e) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$ (f) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^3}$

(g) $\displaystyle\lim_{x\to 4} \frac{x-4}{\sqrt{x^2+9}-5}$ (h) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x\tan x}$ (i) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$

(j) $\displaystyle\lim_{x\to 0} \frac{\cos(ax)+\cosh(ax)-2}{x^4}$ (k) $\displaystyle\lim_{x\to 0} \frac{\sinh x + \sin x - 2x}{x^5}$

A11. We will denote the application of l'Hôpital's rule by means of the following modification of the equals sign, $\stackrel{\text{l'H}}{=\!=}$. The usual equals sign, $=$, means what it has always meant.

(a) This is a zero-divide-zero case. We have,

$$\lim_{x\to\frac{1}{2}} \frac{\cos \pi x}{2x-1} \stackrel{\text{l'H}}{=\!=} \lim_{x\to\frac{1}{2}} \frac{-\pi \sin \pi x}{2} = -\tfrac{1}{2}\pi.$$

(b) Again a zero-divide-zero case. We have,

$$\lim_{x\to 1} \frac{\sin \pi x}{\sqrt{x^2-1}} \stackrel{\text{l'H}}{=\!=} \lim_{x\to 1} \frac{\pi \cos \pi x}{x/\sqrt{x^2-1}} = \lim_{x\to 1} \frac{\pi\sqrt{x^2-1}\cos \pi x}{x} = 0.$$

(c) We have,

$$\lim_{x\to 0} \frac{\cosh ax - 1}{x} \stackrel{\text{l'H}}{=\!=} \lim_{x\to 0} \frac{a \sinh ax}{1} = 0.$$

(d) This is an ∞/∞ case as $x \to \infty$.

$$\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+2x+1} \stackrel{\text{l'H}}{=\!=} \lim_{x\to\infty} \frac{2x+2}{6x+2} \stackrel{\text{l'H}}{=\!=} \lim_{x\to\infty} \frac{2}{6} = \tfrac{1}{3}.$$

This result could also have been found quickly by noting which term in each of the numerator and the denominator is growing the fastest, and then to compare those. Or one could do the following,

$$\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+2x+1} = \lim_{x\to\infty} \frac{1+2x^{-1}+3x^{-2}}{3+2x^{-1}+x^{-2}} = \tfrac{1}{3},$$

where we have divided both the numerator and the denominator by x^2.

(e) We have,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \stackrel{\text{l'H}}{=\!=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \stackrel{\text{l'H}}{=\!=} \lim_{x\to 0} \frac{-\sin x}{6x} \stackrel{\text{l'H}}{=\!=} \lim_{x\to 0} \frac{-\cos x}{6} = -\tfrac{1}{6}.$$

So we had to apply l'Hôpital's rule three times here.

(f) We have,

$$\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^3}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2\sin x\cos x - 2x}{3x^2}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2(\cos^2 x - \sin^2 x) - 2}{6x}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{-4\sin x\cos x}{6} = 0.$$

(g) We have,

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x^2+9}-5} \overset{\text{l'H}}{=} \lim_{x\to 4}\frac{1}{x/\sqrt{x^2+9}} = \tfrac{5}{4}.$$

(h) We have,

$$\lim_{x\to 0}\frac{\sin(x^2)}{x\tan x} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2x\cos(x^2)}{\tan x + x\sec^2 x} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2\cos(x^2) - 4x^2\sin(x^2)}{2(1+\tan x)\sec^2 x} = 1.$$

(i) We have,

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2\tan x\sec^2 x}{6x}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{2\sec^4 x + 4\tan^2\sec^2 x}{6} = \tfrac{1}{3}.$$

(j) We have,

$$\lim_{x\to 0}\frac{\cos(ax)+\cosh(ax)-2}{x^4}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{-a\sin(ax)+a\sinh(ax)-2}{4x^3}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{-a^2\cos(ax)+a^2\cosh(ax)}{12x^2}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{a^3\sin(ax)+a^3\sinh(ax)}{24x}$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0}\frac{a^4\cos(ax)+a^4\cosh(ax)}{24} = \tfrac{1}{12}a^4.$$

(k) We have,

$$
\begin{aligned}
& \lim_{x\to 0} \frac{\sinh x + \sin x - 2x}{x^5} \\
\overset{\text{l'H}}{=\!=} \quad & \lim_{x\to 0} \frac{\cosh x + \cos x - 2}{5x^4} \\
\overset{\text{l'H}}{=\!=} \quad & \lim_{x\to 0} \frac{\sinh x - \sin x}{20x^3} \\
\overset{\text{l'H}}{=\!=} \quad & \lim_{x\to 0} \frac{\cosh x - \cos x}{60x^2} \\
\overset{\text{l'H}}{=\!=} \quad & \lim_{x\to 0} \frac{\sinh x + \sin x}{120x} \\
\overset{\text{l'H}}{=\!=} \quad & \lim_{x\to 0} \frac{\cosh x + \cos x}{120} \quad = \quad \tfrac{1}{60}.
\end{aligned}
$$

---

12. This is not the type of question which I will give in the exam, but I would like to think that it is of interest. Your aim is to use l'Hôpital's rule to find the following three limits:

$$y = \lim_{x\to 0} x^x, \qquad y = \lim_{x\to\infty} x^{1/x}, \qquad \text{and} \qquad y = \lim_{n\to\infty} \Big(1 + \frac{1}{n}\Big)^n.$$

None look like candidates for the use of l'Hôpital's rule, but if one takes the natural logarithm and then rearrange the resulting expressions appropriately, then you'll find that $\ln y$ does indeed take a suitable form. Note that, after you take the logarithm, then (i) the first two cases are of the form of ∞/∞, and (ii) the third case will eventually need the substitution $x = 1/n$ where the limit $x \to 0$ is taken. This third one arises when considering compound interest....more information in the solutions.

A12. These are somewhat unusual ones which will need a little bit of work prior to using l'Hôpital's rule.

(a) Let $y = x^x$ and take natural logarithms of both sides. We get,

$$\ln y = x \ln x.$$

When $x = 0$ the right hand side is of the form, $0 \times \infty$, which can't be determined using l'Hôpital's rule. For the purposes of this question, we may rearrange the right hand side to take the form,

$$\ln y = \frac{\ln x}{1/x}.$$

Now the right hand side is of the form, ∞/∞, as $x \to 0$, and therefore l'Hôpital's rule may be applied:

$$
\begin{aligned}
\lim_{x\to 0} \ln y \quad &= \quad \lim_{x\to 0} \frac{\ln x}{1/x} && \text{rearranged} \\
&\overset{\text{l'H}}{=\!=} \quad \lim_{x\to 0} \frac{1/x}{-1/x^2} && \text{using l'Hôpital} \\
&= \quad \lim_{x\to 0} (-x) && \text{upon tidying up} \\
&= \quad 0.
\end{aligned}
$$

So if $\lim_{x\to 0}\ln y = 0$, then $\lim_{x\to 0} y = 1.$ So in this instance, $0^0 = 1$.

It may be worth using some package to plot the function, $y = x^x$, but don't use negative values of x and don't go higher than $x = 2.5$ because y grows incredibly fast after that. My question now is: can you show that the derivative of x^x is $-\infty$ at $x = 0$?

(b) We take logs again, so let $y = x^{1/x}$ and hence $\ln y = (1/x)\ln x$. On taking limits as $x \to \infty$ we have,

$$\lim_{x\to\infty} \ln y \quad = \quad \lim_{x\to\infty} \frac{\ln x}{x} \quad \stackrel{\text{l'H}}{=\!=} \quad \lim_{x\to\infty} \frac{1/x}{1} \quad = \quad \lim_{x\to\infty} \frac{1}{x} \quad = \quad 0.$$

Hence $\lim\limits_{x\to\infty} x^{1/x} = 1.$ So this tells that $\infty^0 = 1$ in this case, which, by the way, doesn't preclude the possibility of a different limit when different functions are used.

(c) Yet again we take logs. If $y = (1+\frac{1}{n})^n$ then $\ln y = n\ln(1+\frac{1}{n})$. Given the hint, we shall let $x = 1/n$ and hence,

$$\ln y = \frac{1}{x} \times \ln(1+x). \tag{1}$$

The limit, $n \to \infty$, is equivalent to $x \to 0$, and now we have a zero-divide-zero case. Hence,

$$\lim_{x\to 0} \ln y \quad = \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} \quad \stackrel{\text{l'H}}{=\!=} \quad \lim_{x\to 0} \frac{1/(1+x)}{1} \quad = \quad 1.$$

Hence

$$\lim_{n\to\infty} \Big(1 + \frac{1}{n}\Big)^n = e. \tag{2}$$

This result was used in a couple of earlier questions. So we have shown that $1^\infty = e$ in this instance, but there are easy-to-derive alternatives to this final answer. Only a very minor change to the above analysis is need to show that,

$$\lim_{n\to\infty} \Big(1 + \frac{a}{n}\Big)^n = e^a.$$

As a final postscript, we could have also obtained the solution in (2) by expanding the logarithm in (1) using either the Binomial or Taylor's series. For some more discussion, see Example 4.29 in the online notes.
