Derivation of the formula for classifying critical points on a surface

In the lectures we considered the critical points of a surface, $f(x,y)$, and they are defined as being at those values of x and y at which both

$$\frac{\partial f}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0. \tag{1}$$

The classification of the resulting critical points then depends on the value of

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial^2 f}{\partial x \partial y}\Big)^2. \tag{2}$$

The aim of this supplementary handout is to derive this formula.

Let us assume for the sake of simplicity of presentation that a critical point lies at $(x,y)=(0,0)$. Now we may expand $f(x,y)$ as a double Taylor's series about that point, as follows:

$$f(x,y) = f(0,0) + \Big[x f_x(0,0) + y f_y(0,0)\Big] + \tfrac{1}{2}\Big[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\Big] + \cdots. \tag{3}$$

Although we won't be covering Taylor's series in more than one dimension, it is possible to derive the above by first attempting a standard Taylor's series in the x-direction and following that by one in the y-direction. It's worth checking that out.

If we are at a critical point, then $f_x = f_y = 0$. Therefore Eq. (3) reduces to,

$$f(x,y) = f(0,0) + \tfrac{1}{2}\Big[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\Big] + \cdots. \tag{4}$$

If we were to be at a minimum, then the expression in square brackets must be positive for all values of x and y. Likewise, for a maximum, it must always be negative. If the critical point were a saddle point, then it will be sometimes be positive and sometimes negative. So the way we will proceed is to assume at first that we are at a saddle point, and then we'll try to find out in which directions the term in the square bracket is precisely zero. If we can find an expression for those directions, then we have a saddle point, otherwise the critical point is a maximum or a minimum.

Let us take a circular tour about the origin by setting,

$$x = \epsilon\cos\theta \qquad \text{and} \qquad y = \epsilon\sin\theta. \tag{5}$$

Therefore Eq. (4) becomes,

$$f(x,y) = f + \tfrac{1}{2}\epsilon^2\Big[\cos^2\theta f_{xx} + 2\sin\theta\cos\theta f_{xy} + \sin^2\theta f_{yy}\Big] + \cdots, \tag{6}$$

where all functions of x and y on the right hand side are evaluated at $x=y=0$.

If we are at a saddle point, then there will be values of θ for which the term in square brackets in Eq. (6) is zero, and therefore

$$\cos^2\theta f_{xx} + 2\sin\theta\cos\theta f_{xy} + \sin^2\theta f_{yy} = 0. \tag{7}$$

If the angle for which (7) is satisfied does not correspond to where $\cos\theta = 0$, then we may divide both sides of Eq. (7) by $\cos\theta$ to obtain,

$$f_{xx} + 2\tan\theta f_{xy} + \tan^2\theta f_{yy} = 0.(8)$$

This is a quadratic equation for $\tan\theta$, and it has the solution,

$$\tan\theta = \frac{-f_{xy} \pm \sqrt{f_{xy}^2 - f_{xx}f_{yy}}}{f_{yy}}. \tag{9}$$

We note that we could also have divided both sides of (7) by $\sin\theta$, and this would eventually have yielded,

$$\cot\theta = \frac{-f_{xy} \pm \sqrt{f_{xy}^2 - f_{xx}f_{yy}}}{f_{xx}}, \tag{10}$$

which is precisely the same solution.

What happens next depends on the sign of the term inside the square root. When $f_{xx}f_{yy} - f_{xy}^2 < 0$, then the term inside the square root is positive, and hence the expression yields real values for $\tan\theta$. This means that there is a saddle point at $x = y = 0$.

When $f_{xx}f_{yy} - f_{xy}^2 > 0$, then the term inside the square root is negative, and hence there is no real solution for $\tan\theta$. In other words, the term in square brackets in Eq. (6) cannot be zero, and given that it varies continuously as θ varies, it must always be either positive or negative. Further, the fact that $f_{xx}f_{yy} - f_{xy}^2 > 0$ means that f_{xx} and f_{yy} must have the same sign. If both are positive, then the critical point is a minimum, by analogy with the equivalent one dimensional situation. Similarly, a maximum will correspond to when both $f_{xx} < 0$ and $f_{yy} < 0$.

Hence we recover the classification given in the lecture notes, namely that,

$$\begin{array}{ll} f_{xx}f_{yy} - f_{xy}^2 < 0 & \text{Saddle point,} \\ f_{xx}f_{yy} - f_{xy}^2 > 0 & \text{Maximum when } f_{xx} < 0, \\ & \text{Minimum when } f_{xx} > 0. \end{array} \tag{11}$$