University of Bath
Department of Mechanical Engineering

ME20021 Modelling Techniques 2

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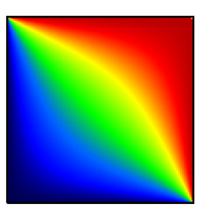
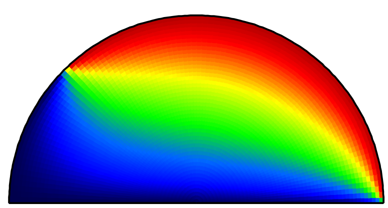
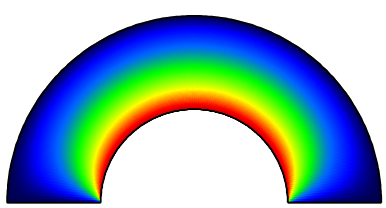
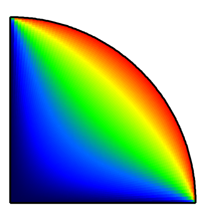
Notes on Analytical Methods for solving PDEs
Fourier Series and Fourier Transforms

Fourier Series

Fourier Transforms

1 Brief outine of the analytical part of this unit

In this unit our aim is to obtain useful analytical solutions of some partial differential equations that arise frequently in science and engineering. In particular we will be concentrating mostly on Fourier's equation, Laplace's equation and the wave equation. The following is a brief description of each. I shall not derive them from first principles because that aspect is well outside of the remit of this unit.

$$\text{Fourier's equation} \qquad \boxed{\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}.} \tag{1}$$

Here, θ is the temperature and, for a one-dimensional domain, θ depends on x and evolves in time. The value, α, is the thermal diffusivity; this is proportional to the thermal conductivity in this way: $\alpha = k/(\rho\, c_p)$ where k is the thermal conductivity, ρ is the density of the conducting medium, and c_p is the specific heat capacity. One practical consequence of the value of α is that, for a given initial temperature profile, the speed of evolution of that profile will be proportional to α. Thus the temperature profile will evolve rapidly when the diffusivity is large, but will evolve slowly when the diffusivity is small. These observations are in accord with our experience.

If we were to replace θ by u, a fluid velocity, and α by the kinematic viscosity, ν, then Fourier's equation now represents how the velocity profile for a unidirectional flow in a channel evolves in time from a given initial velocity profile.

$$\text{Laplace's equation} \qquad \boxed{\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.} \tag{2}$$

Again θ represents the temperature and Eq. (2) describes steady-state heat conduction in two dimensions. One example might be a conducting body with a square cross-section where the four boundaries are maintained at different temperatures.

$$\text{The wave equation} \qquad \boxed{\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2}.} \tag{3}$$

In the wave equation the value, h, typically represents the displacement of a taut string from its equilibrium profile and it then depends on the distance, x, and time. The value, c, is known as the wavespeed (some write *wave speed*) and this is simply because waves travel with this speed! It is natural to visualise these waves as being transverse waves, such as the displacement of a violin string. But they may also represent longitudinal waves — think of expansion and compression waves along a spring — where h represents the density of the material forming the spring. It is now a small step from here to regarding h as being the pressure in a gas (sound waves travel in the same way as longitudinal waves do in a spring) or as compression waves in a solid such as the earth (seismic waves). In these final cases c is the speed of sound within the respective media.

There are extensions and/or higher dimensional analogues of these equations that we do not need to be concerned about in `ME20021` but which I would like to mention just to connect our work here with more realistic situations in real life. These include the,

$$\text{2D Fourier's equation} \qquad \boxed{\frac{\partial \theta}{\partial t} = \alpha \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right),} \tag{4}$$

where an initial two-dimensional temperature profile evolves in time. Hopefully the form of the three dimensional Fourier's equation is now the obvious one!

One important extension is

Poisson's equation $$\boxed{\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=f(x,y).} \tag{5}$$

where $f(x,y)$ is a given function. An example might be the steady-state conduction in a rectangular domain which results from having all four boundaries being maintained at $\theta=0$, and where there is a uniform internal heat generation which is represented by $f(x,y)=-1$. The analogous 3D form could even provide an approximate way of modelling the temperature in a human body where the outer boundary temperature is the ambient temperature, and $f(x,y)$ would take a form which models the heat which is generated during exercise and/or by digesting our food. As with the Laplace's equation above, the replacement of θ by u and α by ν changes the application to one which is a fluid dynamics problem. In this case the solution is equivalent to the flow in a rectangular duct which results from applying a constant pressure gradient along that duct.

We may also model the vibrations of, for example, a drum skin:

The 2D wave equation $$\boxed{\frac{\partial^2 h}{\partial t^2}=c^2\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right).} \tag{6}$$

If the domain were to be a unit square, then this would be equivalent to a square drumskin and the displacement, h, would then need to satisfy $h=0$ on the four boundaries. A three-dimensional version of Eq. (6) could also be used to model pressure waves in the atmosphere or, closer to home, in a room.

At risk of scaring you, a circular drumskin would be best solved using polar coordinates:

The 2D wave equation $$\boxed{\frac{\partial^2 h}{\partial t^2}=c^2\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial\theta^2}\right).} \tag{7}$$

Here r is the radial coordinate and θ is the angular coordinate. For a drumskin of radius R, we would need to use the boundary condition, $h=0$ on $r=R$. We will meet some polar coordinates later, but this particular equation requires Bessel functions to solve it. As a PhD student I recall seeing a two-volume treatise on Bessel functions by a chap called Watson (the version which is currently retailing on Amazon comprises 812 pages); it scared me witless! You may be assured that we won't touch Bessel functions.

Two different techniques will be used to solve these equations: (i) separation of variables followed by **Fourier Series**, and (ii) **Fourier Transforms**. Although there is a technique called *separation of variables* which is used for solving Ordinary Differential Equations (ODEs), the one which we use for solving Partial Differential Equations (PDEs) is different and so, fortunately, it is impossible to confuse these two methods! With regard to Fourier Transforms there is a great deal of similarity between these and Laplace Transforms, but there are some significant differences. From a purely selfish point of view I try to ensure that I don't teach Fourier Transforms to you at the same time as I teach Laplace Transforms to the first years for it gets horribly confusing for me!

In what follows, any technical term which is introduced and which forms a new concept will be typeset in a bold red font, as I have done in the previous paragraph. Other important results or ideas will be given the same treatment.

2 What role do Fourier Series play when solving these PDEs?

Fourier series usually appear in these problems when we are dealing with a finite domain in at least one direction. One example is the time-evolution of the temperature profiles within a one-dimensional heat-conducting domain where the two ends are maintained at the same temperature (a so-called **Dirichlet** boundary condition. Alternatively the two ends could both be subjected to a constant heat flux (a so-called **Neumann** boundary condition) or indeed one of each. The essential observation is that this is a domain of finite width into which one may fit sine (or cosine) waves that already satisfy the boundary conditions.

2.1 Fundamental solutions for Fourier's equation

The aim in this section is to present a fairly general way of finding solutions to the above PDEs. To fix ideas we will solve Fourier's equation, as given in Eq. (1), where the solution will be subject to the conditions that $\theta = 0$ on both $x = 0$ and $x = 1$ and to the initial condition that $\theta = f(x)$, a known function, when $t = 0$. This mathematical description may be translated into the following:

"A solid bar of unit length lies in range $0 \le x \le 1$ and has the temperature profile, $\theta = f(x)$, at $t = 0$, whereupon the ends of the bars have their temperatures changed suddenly to $\theta = 0$; what happens next?"

Even this is too wordy and is better described by the following diagram:

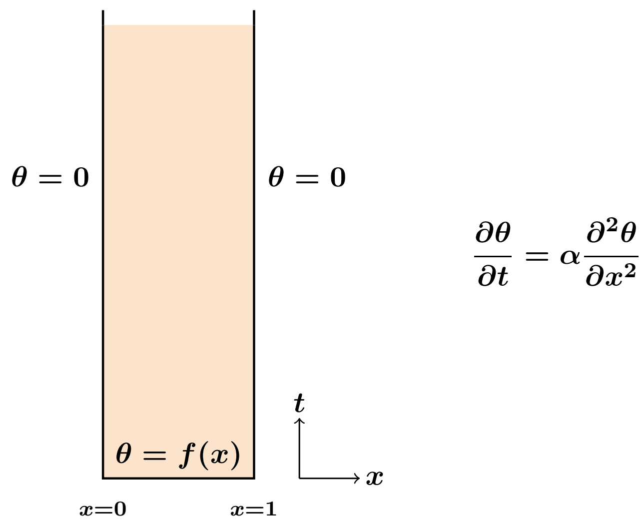


Figure 2.1. A sketch which illustrates the domain within which we shall be solving Fourier's equation, together with the boundary and initial conditions.

We will proceed by means of the following ansatz (i.e. an educated guess):

$$\theta = T(t)\, \sin n\pi x, \tag{8}$$

where n is a positive integer. There are various reasons why this formula needs to be discussed for while it looks as though it has appeared as if by magic there are nevertheless logical reasons why this might be a good thing to use.

Note 1: The t-dependence and the x-dependence in $\theta(x,t)$ have been separated, hence the name of the method, separation of variables.

Note 2 A sine of x has been used. Fourier's equation has a second x-derivative, and therefore if θ is proportional to a sine function of x, then so is its second derivative with respect to x. This means that the result of the substitution is an equation where both sides are proportional to the sine, and hence the sines may be cancelled. This will be done shortly.

Note 3. However, this last observation is also true for cosines. In the present case the analogous cosine ($\cos n\pi x$) will be equal to 1 when $x = 0$, and therefore it does not satisfy the boundary condition that $\theta = 0$ when $x = 0$. Therefore we discount cosines for now and we are stuck with sines because they satisfy $\theta = 0$ at $x = 0$.

Note 4: Ah, but which sines? Well, the ones we have chosen in Eq. (8) are the only ones which fit with the second boundary condition (namely, that $\theta = 0$ when $x = 1$) because $\sin n\pi = 0$ when n is an integer. This choice of sines is illustrated in Fig 2.2, below.

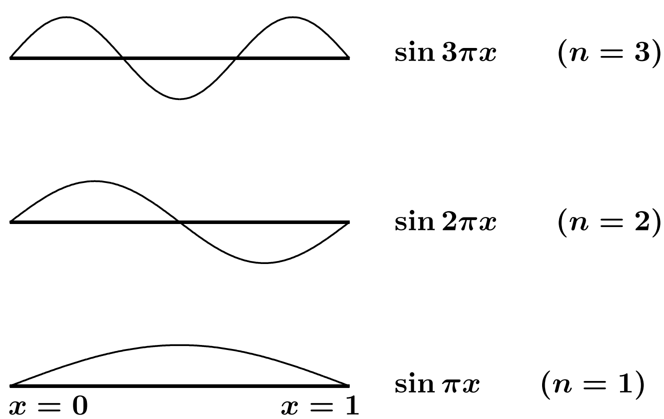


Figure 2.2. The first three modes of the form $\sin n\pi x$ which satisfy zero boundary conditions at $x = 0$ and $x = 1$.

At this point in the analysis the shape of the function $T(t)$ is unknown. Therefore all we can do is to substitute Eq. (8) into Fourier's equation, Eq. (1), and we get

$$\frac{dT}{dt}\cancel{\sin n\pi x} = -\alpha n^2 \pi^2 T \cancel{\sin n\pi x}. \tag{9}$$

The sines cancel, as we expect, leaving us with,

$$\frac{dT}{dt} = -\alpha n^2\pi^2 T, \tag{10}$$

which has the solution,

$$T = Be^{-\alpha n^2\pi^2 t}, \tag{11}$$

where B is an arbitrary constant. Therefore we can say that

$$\boxed{\theta = \underbrace{Be^{-\alpha n^2\pi^2 t}\sin n\pi x}_{\text{fundamental solution}}} \tag{12}$$

is our desired solution at this stage because it satisfies the governing PDE and the two boundary conditions. It is this form which I call a **fundamental solution**.

However, Eq. (12) does not satisfy the given initial condition, namely that $\theta = f(x)$ at $t = 0$, and therefore this single solution is not yet sufficiently general. This difficulty may be resolved by adding together all of the solutions of the form, (12), to give the following,

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2\pi^2 t}\sin n\pi x, \tag{13}$$

where I have introduced the subscript, n, to ensure that all the B-values may take different numerical values should they need to. This simple addition of the fundamental solutions is called **superposition** and it works because the PDE is linear and the boundary conditions are homogeneous (i.e. equal to zero). Now we apply the initial condition; this gives

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x. \tag{14}$$

This looks like a Fourier Series, but technically it is a half-range series consisting of sines. The reason that it is called **half-range** is because the $n = 1$ term corresponds to half a sine wave in the physical domain we are considering; see Fig 2.2. An alternative name, one which we will adopt here, is **Fourier Sine Series**, and this must always be used to mean a half-range series consisting of sines. At this point we would need to apply a formula for the Fourier coefficients, but the formula will be given later after we have considered fundamental solutions for the other two main PDEs that we shall be considering.

We may illustrate a typical solution of Fourier's equation without the use of Fourier Series by using $\theta = \sin\pi x$ as an initial condition. If we refer back to Eq. (12) and let both $n = 1$ and $t = 0$, then we obtain,

$$\sin\pi x = B\sin\pi x \qquad \Longrightarrow \qquad B = 1. \tag{15}$$

Hence the final solution is,

$$\theta = e^{-\alpha\pi^2 t}\sin\pi x, \tag{16}$$

and so the evolution of the temperature field in time is as follows,

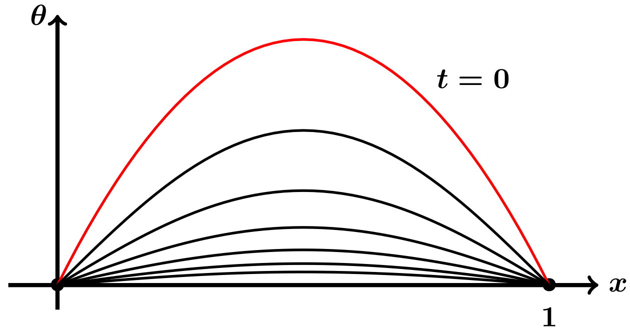


Figure 2.3. Depicting a solution of Fourier's equation. The initial condition is displayed in red. Successive profiles are at a time-interval of $0.05\pi^2/\alpha$.

We see that the initial profile, $\theta = \sin \pi x$, decays in time without a change in the shape. Physically, we see that the heat which was present initially escapes out of the cold boundaries.

2.2 Fundamental solutions for Laplace's equation

We may solve Laplace's equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0, \tag{17}$$

using the same ideas. This equation models a steady two-dimensional temperature distribution in a solid. If we were to consider a semi-infinite strip of solid material, i.e. one which is contained within the domain, $0 \le x \le 1$ and $0 \le y < \infty$, then this is the type of domain where we can make analytical progress. So we'll take the following boundary conditions:

$$y = 0: \quad \theta = f(x), \qquad\qquad y \to \infty: \quad \theta \to 0, \qquad\qquad x = 0, 1: \quad \theta = 0. \tag{18}$$

Physically, these conditions correspond to a strip where the temperature profile of the $y = 0$ end is precisely $f(x)$, and where the infinitely long sides at $x = 0, 1$ are being maintained at $\theta = 0$. I always recommend making a sketch of the configuration being solved, and here it is:

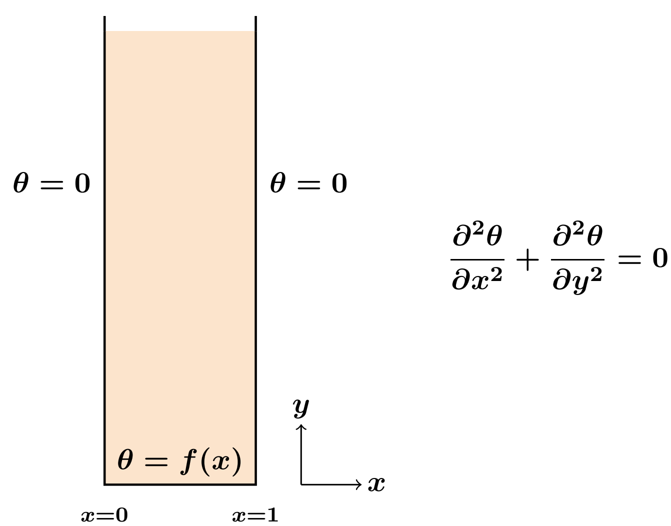


Figure 2.4. Definition sketch for a solution to Laplace's equation.

Apart from the identity of the vertical coordinate this diagram is identical to the one we used for the above Fourier's equation example.

There is one thing missing from this sketch, and that is the boundary condition as $y \to \infty$. Strictly speaking, we need two boundary conditions in the x-direction (we do, at $x = 0, 1$) and two in the y-direction (we don't, just the one at $y = 0$) because we have second order derivatives in each of the two directions. However, we may say that $\theta \to 0$ as $y \to \infty$ on physical grounds. If we suppose that $f(x)$ represents a positive temperature at $y = 0$, then this heat will conduct away from there and be lost via the $\theta = 0$ boundary conditions at $x = 0$ and $x = 1$. So the $y = 0$ boundary is a heat source while the $x = 0, 1$ boundaries are heat sinks. Therefore $\theta \to 0$ as $y \to \infty$.

Now we observe that the strip lies between $x = 0$ and $x = 1$, and that the boundary conditions are both zero. So we may again use sines as part of the separation of variables ansatz, and they will be $\sin n\pi x$, for integer values of n, for exactly the same reasons as for the previous example. Therefore we will substitute

$$\theta = Y(y) \sin n\pi x \tag{19}$$

into Eq. (2), and this yields

$$\frac{d^2Y}{dy^2}\cancel{\sin n\pi x} - n^2\pi^2 Y \cancel{\sin n\pi x} = 0 \qquad \Longrightarrow \qquad \frac{d^2Y}{dy^2} - n^2\pi^2 Y = 0. \tag{20}$$

Equation (20) has the following solutions,

$$Y = Ae^{n\pi y} + Be^{-n\pi y}, \tag{21}$$

where A and B are currently arbitrary constants. Note: This solution was derived using the usual let $Y(y) = e^{\lambda y}$ trick for linear constant-coefficient ODEs.

The substitution of Eq. (21) into Eq. (19) gives the general solution,

$$\theta = \underbrace{\Big[Ae^{n\pi y} + Be^{-n\pi y}\Big] \sin n\pi x.}_{\text{fundamental solution}} \tag{22}$$

We can see immediately that A must be zero because this part of the solution grows exponentially, whereas we need the solution to decay to zero. The solution now reduces to,

$$\theta = Be^{-n\pi y} \sin n\pi x. \tag{23}$$

The final condition to apply is that $\theta = f(x)$ at $y = 0$. To do this, we must first superpose all the possible solutions of the form given in (23):

$$\theta = \sum_{n=1}^{\infty} B_n\, e^{-n\pi y}\, \sin n\pi x, \tag{24}$$

and then apply the boundary condition. This gives,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x, \tag{25}$$

which is again a Fourier Sine Series.

Note that if we were to have a conducting strip orientated in the x-direction with $\theta = 0$ on $y = 0, 1$ and with $\theta = f(y)$ on $x = 0$, then an identical analysis (but with x and y swapped around) would give,

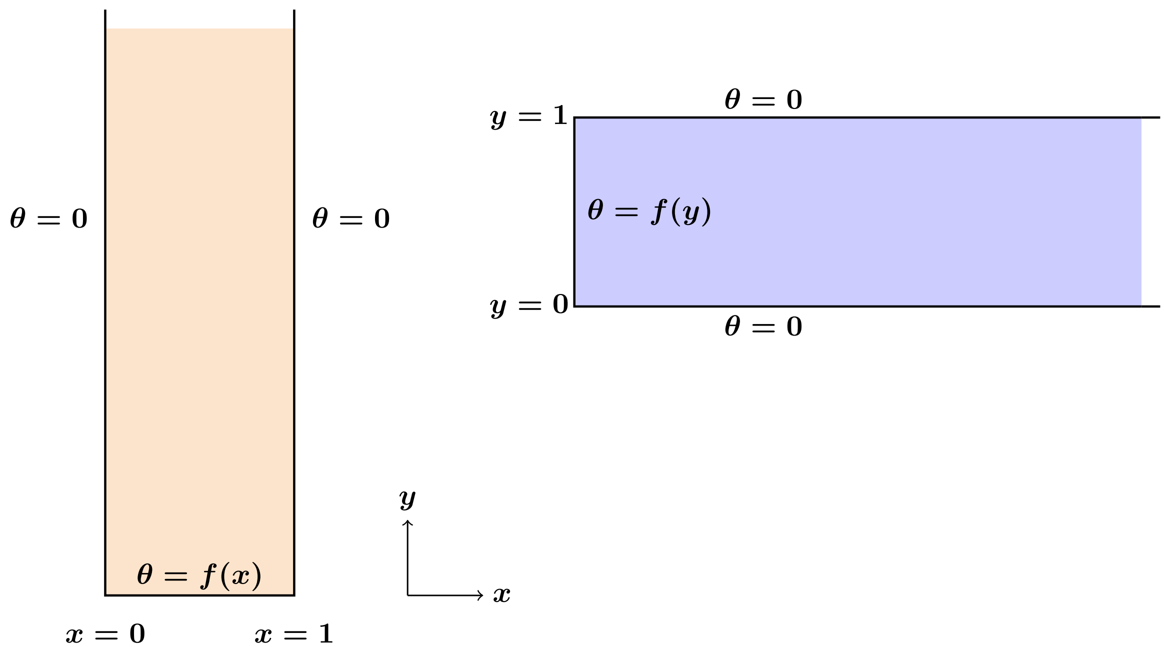
$$\theta = \sum_{n=1}^{\infty} B_n e^{-n\pi x} \sin n\pi y, \tag{26}$$

where the Fourier coefficients are to be determined from the application of the $x = 0$ boundary condition. Therefore we have

$$f(y) = \sum_{n=1}^{\infty} B_n \sin n\pi y, \tag{27}$$

from which the Fourier coefficients, B_n, will be calculated. Do compare this with Eq. (25).

Here's a final illustration of how similar the solutions are for these two strip orientations:



$$\boxed{\theta = \sum_{n=1}^{\infty} B_n e^{-n\pi y} \sin n\pi x.} \qquad \boxed{\theta = \sum_{n=1}^{\infty} B_n e^{-n\pi x} \sin n\pi y.}$$

Figure 2.5. The same mathematical problem!

2.3 Fundamental solutions for the wave equation

We will solve the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{28}$$

subject to the boundary conditions, $y = 0$ at $x = 0$ and $x = 1$, and the initial conditions, $y = f(x)$ and $\partial y/\partial t = 0$ at $t = 0$. The initial conditions are equivalent to plucking a violin string in the sense that the string has an initial deformation profile and it is then released from rest; see Fig. 2.6 below. The boundary conditions state that there is zero displacement at the two ends, which is quite natural, or, rather, quite essential for a properly functioning violin string.

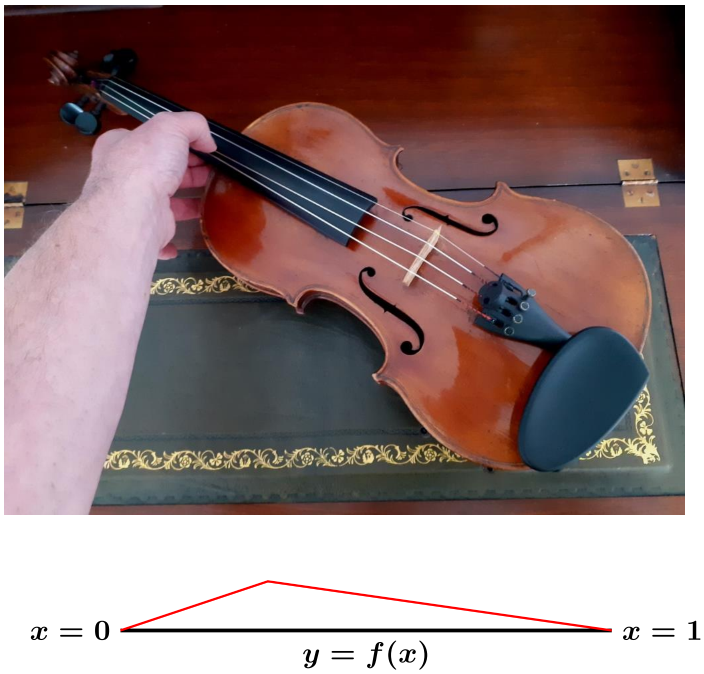


Figure 2.6. Displaying a typical piecewise-linear initial profile. Here we see a rather large piecewise-linear displacement of a violin A-string. I am glad to report that no string was broken during the creation of this photograph.

As with the previous two cases we will factor out a $\sin n\pi x$ dependence by setting,

$$y(x,t) = T(t) \sin n\pi x, \tag{29}$$

where n takes positive integer values, and therefore the equation reduces to

$$\frac{d^2 T}{dt^2} = -n^2\pi^2 c^2 T \qquad \text{or} \qquad \frac{d^2 T}{dt^2} + n^2\pi^2 c^2 T = 0 \tag{30}$$

after cancellation of the sines. The solution for T is

$$T = A\cos n\pi ct + B \sin n\pi ct, \tag{31}$$

and therefore the fundamental solution which we are seeking is,

$$y = \underbrace{\Big[A\cos n\pi ct + B\sin n\pi ct\Big] \sin n\pi x.}_{\text{fundamental solution}} \tag{32}$$

Thus far we have written down a solution of the PDE which satisfies both the boundary conditions at $x = 0$ and $x = 1$.

We now need to superpose all the fundamental solutions and then to apply the initial conditions. After superposition we have,

$$y = \sum_{n=1}^{\infty} \Big[A_n \cos n\pi ct + B_n \sin n\pi ct \Big] \sin n\pi x. \tag{33}$$

It is easier to apply the condition, $\partial y/\partial t = 0$ at $t = 0$, first because it is of "something $= 0$" form! The quickest way to do this is to observe that $\cos n\pi t$ already has a zero derivative at $t = 0$, but $\sin n\pi t$ does not, and therefore we must suppress the latter by setting $B_n = 0$ for all values of n. This may also be shown in the usual way by first finding $\partial y/\partial t$ and then substituting $t = 0$. The solution now reads,

$$y = \sum_{n=1}^{\infty} A_n \cos n\pi ct \sin n\pi x. \tag{34}$$

Application of the initial condition that $y = f(x)$ at $t = 0$ yields the Fourier Sine Series,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x. \tag{35}$$

Once the A_n values have been obtained, we may substitute them into Eq. (34) and that will complete the solution for y.

Note: given that the Fourier coefficients for a sine series are, by convention, denoted by a B, we may replace A_n by B_n at this point should we find ourselves with an A_n as opposed to the traditional B_n. Therefore we shall replace Eq. (35) by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x. \tag{36}$$

2.4 Some comments

Note 1. It is important to point out that the factoring out of appropriate sines, which is the way in which we have been applying the separation of variables method, is *not* the technique which is given in textbooks. You have been warned!

Therefore I have placed a description of that textbook method in §9 for your interest and information and, more importantly, to avoid you freaking out when you consult a library shelfful of textbooks and not find my method. The reason that I have adopted the process described above is that it is much quicker and more intuitive for the types of equation we are solving in `ME20021`.

Note 2. All of the above solutions have corresponded to systems in which the physical domain of interest has a unit length. Thus the conducting 1D solid bar (Fourier's equation), the 2D conducting strip (Laplace's equation) and the taut string (the wave equation) all occupy the range $0 \le x \le 1$.

But what happens to the fundamental solutions when the strip/string have length, d, say, where $0 \leq x \leq d$? Well, in this case we could have $0 \leq (x/d) \leq 1$, and therefore we should use the function, $\sin(n\pi x/d)$, in the separation of variables ansatz when the boundary conditions are zero at $x = 0$ and $x = d$. An easy way of remembering the form of this sine wave is to realise that $0 \leq (x/d) \leq 1$, and this implies that $\sin n\pi(x/d)$ should be used.

So if we now wish to solve Fourier's equation in the range $0 \leq x \leq d$ with $\theta = 0$ on $x = 0, d$, then Eq. (8) should be replaced by

$$\theta = T(t) \sin(n\pi x/d). \tag{37}$$

The function, T, now satisfies the equation,

$$\frac{dT}{dt} = -\alpha \frac{n^2\pi^2}{d^2} T, \tag{38}$$

and has the solution,

$$T = e^{-\alpha n^2\pi^2 t/d^2}. \tag{39}$$

Therefore the full solution, without the initial condition having been applied, is

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2\pi^2 t/d^2} \sin(n\pi x/d). \tag{40}$$

An alternative derivation of which sine to use is the following. Let the sine be $\sin \sigma x$ where σ is presently unknown and needs to be found. When $x = d$ we have $\sin \sigma d$ which must be zero in order to satisfy the boundary condition there. Therefore $\sigma d = n\pi$, since $\sin n\pi = 0$ when n is an integer. Therefore we find that $\sigma = n\pi/d$, and finally the sine that we need is $\sin(n\pi x/d)$, as before.

Note 3. If we have the situation where the strip/string occupies the region, $-d \leq x \leq d$, then the simplest tactic is to redefine the x–coordinate. We would set $\tilde{x} = x + d$, which means that $0 \leq \tilde{x} \leq 2d$. Therefore we would need to use $\sin(n\pi\tilde{x}/2d)$. I will leave you to prove that,

$$\frac{\partial^2\theta}{\partial x^2} \text{ is equal to } \frac{\partial^2\theta}{\partial \tilde{x}^2},$$

which means that the PDE won't change its appearance apart from that tilde. However, I will not give you any examples where this needs to be done.

3 Definitions of the various Fourier series

Here we shall state the definitions of the Fourier series (plural) in some of its many guises. There will be four of these which may be used in ME20021. That sounds scary but the definitions of each are remarkably similar, and should you find yourself in an invigilated exam then the definition of the appropriate one will be given on the paper itself. We will also discuss convergence briefly.

3.1 Fourier Series

This is the familiar one from the year 1 Mathematics units.

If the function $f(x)$ has a period equal to d and has been defined explicitly in the range, $0 \le x \le d$, then the Fourier Series is

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big[A_n \cos(2n\pi x/d) + B_n \sin(2n\pi x/d)\Big] \tag{41}$$

where

$$A_0 = \boxed{\frac{2}{d}\int_0^d f(x)\,dx,} \qquad A_n = \boxed{\frac{2}{d}\int_0^d f(x)\cos(2n\pi x/d)\,dx,}$$

and

$$B_n = \boxed{\frac{2}{d}\int_0^d f(x)\sin(2n\pi x/d)\,dx.} \tag{42}$$

Note that the range of integration is one period, and therefore the actual limits used will depend on the range over which $f(x)$ is defined explicitly. So if $f(x)$ has been defined explicitly over the range, $-\tfrac{1}{2}d \le x \le \tfrac{1}{2}d$, then the only change in Eq. (42) is that the limits of the integrals are now from $x = -\tfrac{1}{2}d$ to $x = \tfrac{1}{2}d$, as follows,

$$\boxed{A_0 = \frac{2}{d}\int_{-d/2}^{d/2} f(x)\,dx,} \qquad \boxed{A_n = \frac{2}{d}\int_{-d/2}^{d/2} f(x)\cos(2n\pi x/d)\,dx,}$$

and

$$\boxed{B_n = \frac{2}{d}\int_{-d/2}^{d/2} f(x)\sin(2n\pi x/d)\,dx.} \tag{43}$$

Note also that the $n = 1$ cosine is $\cos(2\pi x/d)$ and this executes one full cosine wave over the period, d. Likewise, the $n = 1$ sine, $\sin(2\pi/d)$, executes one full sine wave of the period, d.

We will need this Fourier Series when solving Laplace's equation in polar coordinates in a circular or even an annular domain.

The following Figure shows how successive partial sums converge to the given function that they are meant to model/replace as the number of terms, m, increases. Here we use $f(x) = x$ in the range $0 \le x \le 1$ so that the function has a period, $d = 1$. Substitution into Eq. (42) yields $A_n = 0$ except for $A_0 = 1$, and $B_n = -1/n\pi$. Hence the Fourier Series is

$$f(x) = \tfrac{1}{2} - \sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n\pi}. \tag{44}$$

In Fig. 3.1, m terms means the constant plus $\sum_{n=1}^{m}$ in Eq. (44); this is the m^{th} partial sum.

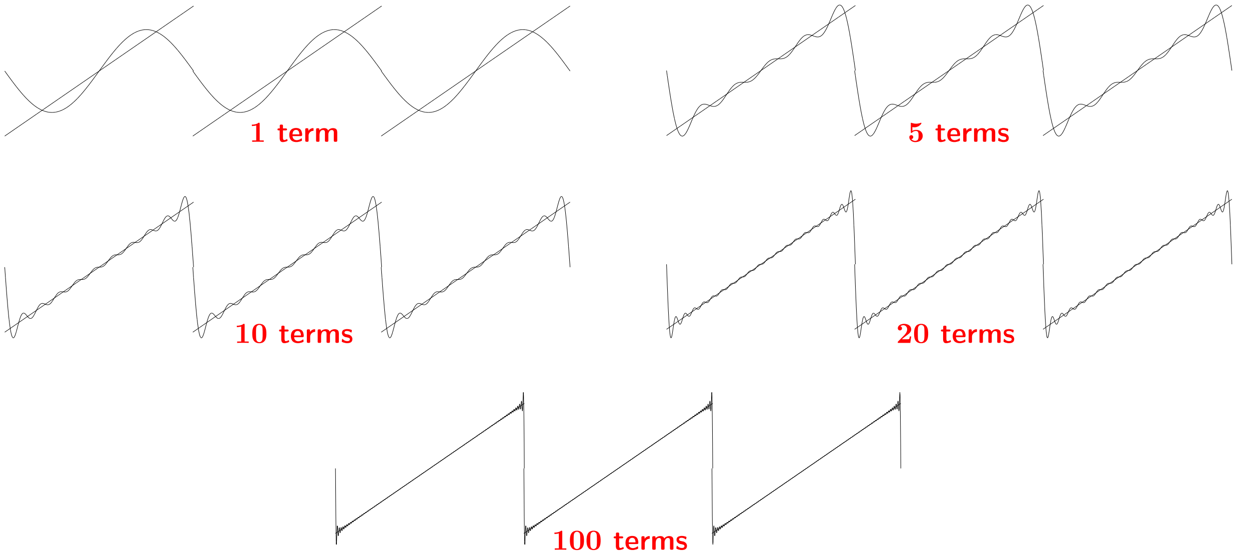


Figure 3.1. Showing the convergence characteristics of the full Fourier Series. Showing three periods of data.

When only one term is used, then the original function is poorly modelled by $\frac{1}{2} - \sin 2\pi x$.

The fifth partial sum is when we retain only the first five terms in the summation, i.e. we have

$$\begin{aligned} f(x) &\simeq \frac{1}{2} - \sum_{n=1}^{5} \frac{\sin 2n\pi x}{n\pi} \\ &= \frac{1}{2} - \frac{\sin 2\pi x}{\pi} - \frac{\sin 4\pi x}{2\pi} - \frac{\sin 6\pi x}{3\pi} - \frac{\sin 8\pi x}{4\pi} - \frac{\sin 10\pi x}{5\pi}. \end{aligned}$$

This truncated Fourier Series entwines $f(x)$ fairly well, and it is interesting to note that the discontinuity in $f(x)$ is modelled by a sharply-descending curve.

When the number of terms increases to 10, then to 20 and finally to 100, then $f(x)$ is approximated increasingly well. The discontinuity is modelled by an increasingly steep curve as continuous functions (sines here) struggle to mimic a discontinuous function. There remains an overshoot immediately before and after the discontinuity of roughly 9% of the drop, but this overshoot region decreases in width as the number of terms increases; these features are called Gibb's phenomenon.

3.2 Fourier Sine Series

This is the half-range series consisting of sines which we have met above in §2. It is used when solving equations where the function is zero on both of the boundaries. Such conditions are known as Dirichlet conditions.

If the function $f(x)$ is defined in the range, $0 \le x \le d$, then the Fourier Sine Series is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/d) \tag{45}$$

where the Fourier coefficients are given by,

$$\boxed{B_n = \frac{2}{d}\int_0^d f(x)\sin(n\pi x/d)\,dx.} \tag{46}$$

Note that the number by which the integral is multiplied is the same as for standard Fourier Series, namely 2 divided by the length of interest. However, the sine terms are slightly different, and the $n = 1$ sine is $\sin(\pi x/d)$ which executes *half* a sine wave in the given interval, $0 \le x \le d$.

We will consider the function, $f(x) = x$, again but it is now defined solely on the interval, $0 \le x \le 1$, rather than being periodic with a period equal to 1 as we had for the full Fourier Series earlier. If we use this function in Eq. (46) with $d = 1$ then we obtain,

$$B_n = \frac{2(-1)^{n+1}}{n\pi}. \tag{47}$$

Hence the Fourier Sine Series representation of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\sin n\pi x}{n\pi}. \tag{48}$$

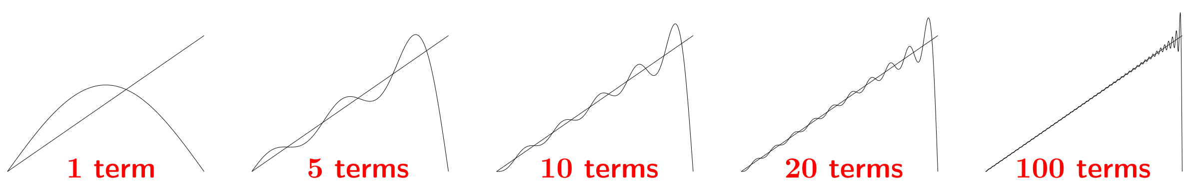


Figure 3.2. Showing the convergence characteristics of a Fourier Sine Series.

Figure 3.2 shows how the Fourier Sine Series representation of this $f(x)$ improves with the number of terms which are retained in the partial sums. It might seem to be a little strange that the value of the Fourier Sine Series is zero at $x = 1$, but that is because we have used a series which is composed of sines all of which are zero there — this choice of sines will have been dictated by the boundary conditions of the PDE which is being solved. We also see Gibb's phenomenon near to $x = 1$ where the series struggles to model a function which is nonzero at $x = 1$.

3.3 Fourier Cosine Series

This is the half-range series consisting of cosines, and it is used when solving equations where the *derivative* of the function is zero on the boundaries. Such conditions are known as Neumann conditions. We will tackle such problems later.

If the function $f(x)$ is defined in the range, $0 \le x \le d$, then the Fourier Cosine Series is

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/d) \tag{49}$$

where

$$\boxed{A_0 = \frac{2}{d}\int_0^d f(x)\,dx,} \qquad \boxed{A_n = \frac{2}{d}\int_0^d f(x)\cos(n\pi x/d)\,dx.} \tag{50}$$

As with the Fourier Sine Series, the $n = 1$ cosine is $\cos(\pi x/d)$ which executes half a cosine wave in the given interval, $0 \le x \le d$. The value, $\frac{1}{2}A_0$, is the mean value of the function, $f(x)$, just as it is in the full Fourier Series above. These are illustrated in the following sketches of some mode shapes.

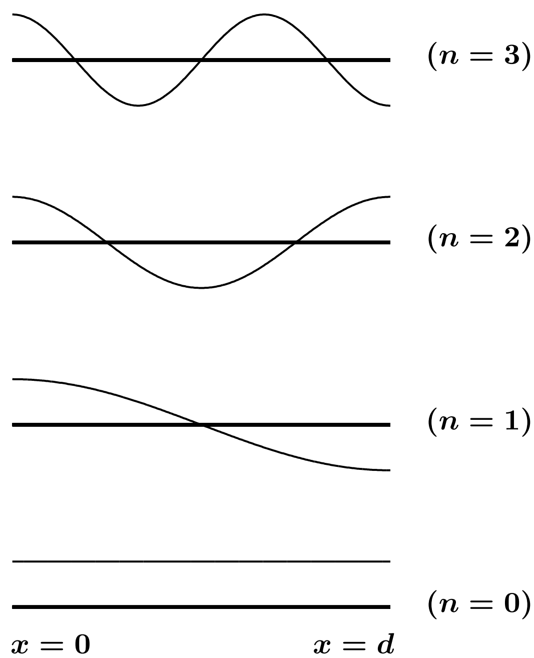


Figure 3.3. The first few modes which are used for a Fourier Cosine Series.

As an example of a Fourier Cosine Series we may again choose the function, $f(x) = x$ in the range $0 \le x \le 1$. Using this and $d = 1$ in Eqs. (50) then we obtain,

$$A_0 = 1, \qquad A_n = -\frac{4}{n^2\pi^2} \quad (n \text{ odd}), \qquad A_n = 0 \quad (n \text{ even}). \tag{51}$$

Therefore the Fourier Cosine Series of $f(x)$ is,

$$f(x) = \tfrac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x. \tag{52}$$

Note that the summation here is over odd values of n only.

The convergence of the Fourier Cosine Series for this $f(x)$ is shown below:

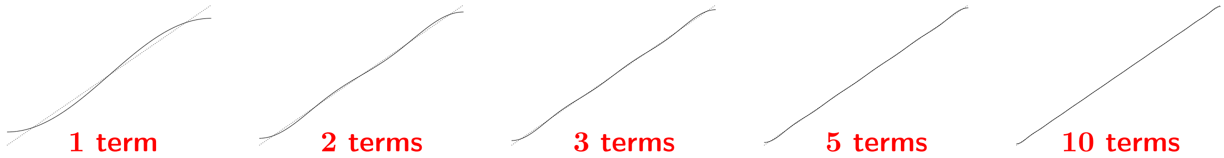


Figure 3.4. Showing the convergence characteristics of a Fourier Cosine Series.

Compared with the full Fourier Series and the Fourier Sine Series we see that fewer terms are required to obtain an excellent approximation to $f(x)$. There are technical reasons for this which I won't discuss here, but there are functions where the Fourier Sine Series converges faster than the Fourier Cosine Series. In the present case the slopes of the cosine terms at $x = 0$ and $x = 1$ are zero, but the slopes of $f(x)$ at $x = 0$ and $x = 1$ are nonzero. So it seems to be the case that such a mismatch between the slopes is less serious than a mismatch between the values.

3.4 A quarter-range Fourier Sine Series

Such a series will arise when solving a PDE where the solution has to satisfy a zero boundary condition at $x = 0$, but to have a zero derivative at $x = d$. The first quarter of a sine wave is an example of a function which has these properties. In this range, the sine wave will be $\sin(\pi x/2d)$, where the argument to the function, namely $\pi x/2d$ varies from 0 to $\pi/2$ as x varies from 0 to d. The other sine waves with this property are $\sin(n\pi x/2d)$ where n takes odd integer values only. Such modes are shown in the following figure.

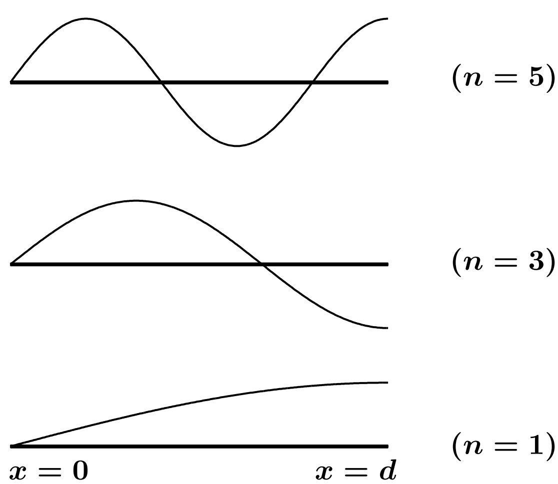


Figure 3.5. The first three modes which are used for a quarter-range Fourier Sine Series.

If the function $f(x)$ is defined in the range, $0 \leq x \leq d$, then the quarter-range Fourier Sine Series is given by,

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B_n \sin(n\pi x/2d) \tag{53}$$

where

$$\boxed{B_n = \frac{2}{d}\int_0^d f(x)\sin(n\pi x/2d)\,dx.} \tag{54}$$

If, as before, we choose to use $f(x) = x$ in the range, $0 \leq x \leq 1$, then the quarter-range Fourier Series coefficients are obtained using Eq. (54) with $d = 1$. We obtain

$$B_n = \frac{8\sin(\frac{1}{2}n\pi)}{n^2\pi^2}. \tag{55}$$

Therefore the quarter-range series for $f(x)$ is,

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8\sin(\frac{1}{2}n\pi)}{n^2\pi^2}\sin(\tfrac{1}{2}n\pi x). \tag{56}$$

The Fourier coefficient includes the rather strange term, $\sin(\frac{1}{2}n\pi)$. A more familiar trigonometric term of the same kind is $\cos n\pi$ which, if one were to sketch it for integer values of n, may be shown easily to be the same as $(-1)^n$. In the present quarter-range Fourier Series context we are confined to odd values of n. So when n follows the sequence, $1,\ 3,\ 5,\ 7\cdots$, then $\sin(\frac{1}{2}n\pi)$ follows the sequence, $1,\ -1,\ 1,\ -1\cdots$. This is difficult to express in a different way using n unless one wished to use fractions: $(-1)^{(n-1)/2}$. A fraction-free alternative may be found by redefining the summation counter, n, using $n = 2m+1$, and then it would be $(-1)^m$. Thus the quarter range series given in Eq. (56) may be rewritten as,

$$f(x) = \sum_{m=0}^{\infty} \frac{8(-1)^m}{(2m+1)^2\pi^2}\sin\Big(\tfrac{1}{2}(2m+1)\pi x\Big). \tag{57}$$

Finally, the behaviour of the first few partial sums may now be seen:

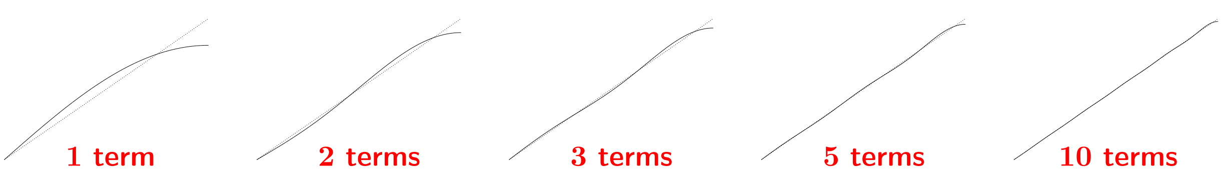


Figure 3.6. The convergence characteristics of a quarter-range Fourier Series.

3.5 Some comments.

In the four examples of Fourier series given above, we used the same function, $f(x) = x$, to illustrate the use of the various Fourier Series to approximate that function. While the full Fourier Series representation is valid for the extension of $f(x)$ into a periodic function with a unit period, and hence it applies in $-\infty < x < \infty$, the other three are valid only for the domain of interest, namely $0 \leq x \leq 1$. However, all four of these different Fourier Series converge to $f(x) = x$ within $0 \leq x < 1$.

Many will recall d'Alembert's test for the convergence of series which was covered in Year 1. It was concerned with (i) whether or not a numerical series converges and (ii) the determination of the radius of convergence of a power series. These ideas should not be confused with what is happening in the Fourier world. Fourier series converge. The primary concern here is with the speed of convergence, and this will depend on the power of n in the denominator of the Fourier coefficients, A_n and B_n.

Note: Section 2 was concerned with the method of separation of variables and how it may be used to begin the solution of certain PDEs. This section, Section 3, has been concerned with the definition of four different types of Fourier Series, and their has been illustrated. In practice we will need both of these techniques. Sections 4 to 8 will provide quite a few examples of this.

4 Fourier Sine Series solutions of some PDEs

4.1 Fourier Sine Series Solutions of Fourier's equation.

First, we recall that Fourier's equation is,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}. \tag{58}$$

Example 4.1. We will solve this PDE subject to $\theta = 0$ on $x = 0$ and $x = 1$ and with the initial condition, $\theta = x(1 - x)$ at $t = 0$. Strictly, I would always advise that a quick sketch of the domain is made together with the boundary and initial conditions; this always assists the writing down of the separation-of-variables ansatz. But we did this many times in §2, and so I shall omit the sketch here.

Given that $\theta = 0$ on both $x = 0$ and $x = 1$ we substitute

$$\theta = T(t)\sin(n\pi x) \tag{59}$$

into Fourier's equation, where n is a positive integer. Hence,

$$\begin{aligned} & \frac{dT}{dt}\cancel{\sin n\pi x} = -\alpha n^2\pi^2 T\cancel{\sin n\pi x} \\ \Longrightarrow \qquad & \frac{dT}{dt} = -\alpha n^2\pi^2 T \\ \Longrightarrow \qquad & T = Be^{-\alpha n^2\pi^2 t}. \end{aligned} \tag{60}$$

On reconstructing θ from (59), and then superposing all the possible solutions, we have

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2\pi^2 t}\sin(n\pi x). \tag{61}$$

On applying the given initial condition, $\theta = x(1-x)$ at $t=0$, we have

$$x - x^2 = \sum_{n=1}^{\infty} B_n \sin(n\pi x), \tag{62}$$

which is a Fourier Sine Series.

The values of the Fourier coefficients are given by using the definition of the Fourier Sine Series integral in Eq. (46) with $d=1$. Therefore,

$$\begin{aligned}
B_n &= 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx \\
&= 2\Bigg[\underbrace{\Big(x-x^2\Big)}_{D_0}\underbrace{\Big(-\frac{\cos n\pi x}{n\pi}\Big)}_{I_1} - \underbrace{\Big(1-2x\Big)}_{D_1}\underbrace{\Big(-\frac{\sin n\pi x}{n^2\pi^2}\Big)}_{I_2} + \underbrace{\Big(-2\Big)}_{D_2}\underbrace{\Big(\frac{\cos n\pi x}{n^3\pi^3}\Big)}_{I_3}\Bigg]_0^1 \\
&= -\frac{4}{n^3\pi^3}\Big[\cos n\pi x\Big]_0^1 \\
&= -\frac{4}{n^3\pi^3}\Big[\cos n\pi - \cos 0\Big] \qquad = \qquad -\frac{4}{n^3\pi^3}\Big[(-1)^n - 1\Big] \\
&= \frac{8}{n^3\pi^3} \quad \text{for } n \text{ odd,} \qquad \text{or} \qquad 0 \quad \text{for } n \text{ even.}
\end{aligned} \tag{63}$$

Therefore the final solution is,

$$\boxed{\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} e^{-\alpha n^2\pi^2 t}\sin n\pi x.} \tag{64}$$

Figure 4.1 shows how the initial temperature profile, $\theta = x(1-x)$, evolves in time.

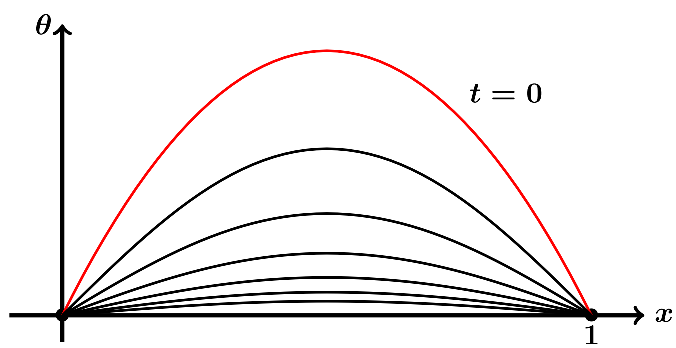


Figure 4.1. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive profiles are at a time-interval of $0.05\pi^2/\alpha$.

Note: Although the initial condition is a quadratic in x it is difficult to distinguish its shape visually from half a sine wave. The reason for that is that the magnitude of the second Fourier coefficient (i.e. the $n=3$ term) is $\frac{1}{27}$th that of the first term, and it is therefore not far from being negligible. As time progresses, the second and subsequent terms decay much more rapidly than the first does, and the temperature profiles are then almost indistinguishable from the leading sine term in Eq. (64).

Example 4.2. The only difference between this example and Example 4.1 is that the initial condition is $\theta = x$ at $t = 0$. This is the function which was used as a specific example of an $f(x)$ in §2. Therefore everything which lies between Eq. (59) and (61) ought to be repeated here, but we shall omit this for the sake of brevity and resume that analysis at the point where the initial condition is applied.

Beginning at Eq. (61), which is

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2\pi^2 t} \sin(n\pi x),$$

we let $t = 0$. Hence,

$$x = \sum_{n=1}^{\infty} B_n \sin(n\pi x), \tag{65}$$

which is again a Fourier Sine Series. On applying the formula for the Fourier Sine Series coefficents with $d = 1$ we obtain the following analysis:

$$\begin{aligned} B_n &= 2\int_0^1 x \sin(n\pi x)\, dx \\ &= 2\Big[\underbrace{\Big(\, x \,\Big)}_{D_0}\underbrace{\Big(-\frac{\cos n\pi x}{n\pi}\Big)}_{I_1} - \underbrace{\Big(\, 1 \,\Big)}_{D_1}\underbrace{\Big(-\frac{\sin n\pi x}{n^2\pi^2}\Big)}_{I_2}\Big]_0^1 \\ &= -\frac{2}{n\pi}\Big[x\cos n\pi x\Big]_0^1 \quad = \quad -\frac{2\cos n\pi}{n\pi} \quad = \quad \frac{2(-1)^{n+1}}{n\pi} \end{aligned} \tag{66}$$

Therefore the final solution is,

$$\boxed{\theta = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-\alpha n^2\pi^2 t} \sin n\pi x,} \tag{67}$$

which may be compared with the solution quoted in Eq. (48).

The evolution of the temperature profile with time is given below.

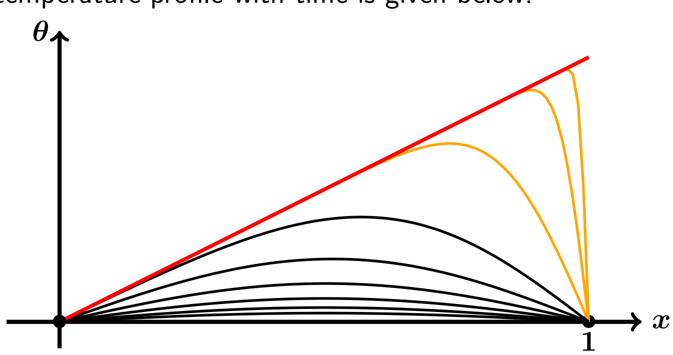


Figure 4.2. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05\pi^2/\alpha$. Orange profiles correspond to $0.0001\pi^2/\alpha$, $0.001\pi^2/\alpha$ and $0.01\pi^2/\alpha$.

It is very clear that the discontinuity in the temperature profile at $x = 1$ when $t = 0$ (i.e. the given initial profile has $\theta = 1$ there, but the boundary condition has $\theta = 0$. Thus the effect of the cold boundary diffuses inwards very rapidly at early times, although this may be phrased differently as: the interior heat is lost rapidly from the cold boundary at early times. The three orange profiles correspond to the earliest times. Thereafter the different exponential decay rates for the different modes means that the $n = 1$ mode, $\sin \pi x$, dominates. This may be seen easily because the later profiles again look like half a sine wave.

Example 4.3. We will now consider the piecewise linear initial condition,

$$\theta = f(x) = \begin{cases} x & (0 \le x \le \frac{1}{2}), \\ 1 - x & (\frac{1}{2} \le x \le 1). \end{cases} \tag{68}$$

As we did for Example 4.2, we may dispense with the separation of variables part of the analysis because it is exactly the same as for Example 4.1. So we'll begin with Eq. (61):

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2 \pi^2 t} \sin(n\pi x).$$

We have defined $f(x)$ to be the piecewise-linear initial profile given in Eq. (68), which one may see in Fig. 4.33 below as the red line. The application of the initial condition yields,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x), \tag{69}$$

where B_n is given using the usual Fourier Sine Series formula. This gives,

$$\begin{aligned} B_n &= 2\int_0^1 f(x) \sin n\pi x \, dx \\ &= 2\int_0^{1/2} x \, \sin n\pi x \, dx + 2\int_{1/2}^1 (1 - x) \, \sin n\pi x \, dx \\ &= \text{Some detailed integrations by parts} \\ &= \frac{4 \sin(\frac{1}{2} n\pi)}{n^2 \pi^2}. \end{aligned} \tag{70}$$

The integrations by parts that have been omitted don't take long but are just a little annoying. However, we now have B_n and therefore the final solution is,

$$\theta = \sum_{n=1}^{\infty} \frac{4 \sin(\frac{1}{2} n\pi)}{n^2 \pi^2} e^{-\alpha n^2 \pi^2 t} \sin n\pi x. \tag{71}$$

The evolution of this initial temperature profile is illustrated below.

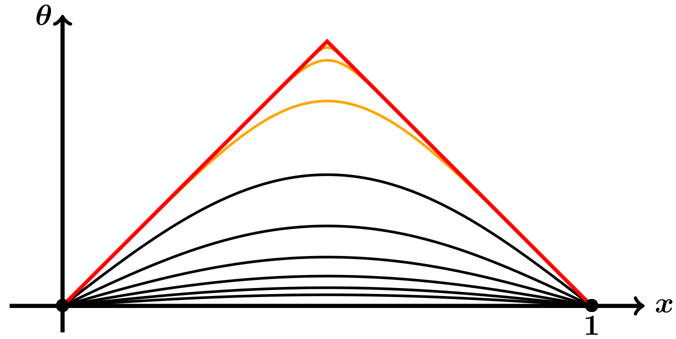


Figure 4.3. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05\pi^2/\alpha$. Orange profiles correspond to $0.0001\pi^2/\alpha$, $0.001\pi^2/\alpha$ and $0.01\pi^2/\alpha$.

It is interesting to note that when a linear temperature profile such as $\theta = x$ or $\theta = 1 - x$ is substituted into Fourier's equation, then $\partial\theta/\partial t = 0$ because $\partial^2\theta/\partial x^2 = 0$ for a linear profile. So θ does not evolve in time! Such an observation might make one wonder how the present initial profile does in fact evolve. This is because of the discontinuous slope of the initial profile at $x = \frac{1}{2}$. Heat is being lost at both $x = 0$ and $x = 1$ because of the nonzero slopes of the initial profile at those points, and ultimately that is why the temperature at $x = \frac{1}{2}$ begins to decrease initially even if it doesn't elsewhere. But as time progresses, yes you've guessed it, the $n = 1$ mode decays the slowest and eventually dominates. Thus the later temperature profiles are essentially half of a sine wave in shape.

Note: The Fourier coefficient contains the term, $\sin\frac{1}{2}n\pi$, something which we have seen before but it was in the context of a quarter range series where n was constrained to taking odd values. Here, n involves all the positive integers. However, when n follows the sequence, $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \cdots$, the term, $\sin\frac{1}{2}n\pi$ takes the values, $1,\ 0,\ -1,\ 0,\ 1,\ 0\ \cdots$. Therefore it is the analysis itself which tells us that only odd values of n are needed. Earlier, when we were considering a quarter-range series, we changed the summation counter using $n = 2m + 1$ and we may do so again here. This eventually leads to the following alternative form of the final solution:

$$\theta = \sum_{m=0}^{\infty} \frac{4(-1)^m}{(2m+1)^2\pi^2} e^{-\alpha(2m+1)^2\pi^2 t} \sin(2m+1)\pi x. \tag{72}$$

Well, that is more compact in some ways, but I think that I still prefer Eq. (71).

Example 4.4. This final case involves a discontinuous asymmetric initial condition but again with $\theta = 0$ boundary conditions at $x = 0$ and $x = 1$. The initial condition is,

$$\theta = f(x) = \begin{cases} 0 & (0 \le x \le \frac{1}{2}), \\ 1 & (\frac{1}{2} \le x \le \frac{3}{4}), \\ 0 & (\frac{3}{4} \le x \le 1). \end{cases} \tag{73}$$

One may take a peek at this initial condition in Fig. 4.4 where it is represented by the red line.

Yet again, we begin the present analysis at the end of the separation-of-variables stage represented by Eq. (61), although all of the separation-of-variables analysis in an exam will need to be included!

We have,

$$\theta = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2 \pi^2 t} \sin(n\pi x), \tag{74}$$

and the application of the initial condition at $t = 0$ yields,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x), \tag{75}$$

where the Fourier coefficients are given by,

$$\begin{aligned}
B_n &= 2\int_0^1 f(x)\,\sin n\pi x\,dx \\
&= \cancel{2\int_0^{1/2} 0\,\sin n\pi x\,dx} + 2\int_{1/2}^{3/4} 1\,\sin n\pi x\,dx + \cancel{2\int_{3/4}^{1} 0\,\sin n\pi x\,dx} \\
&= 2\int_{1/2}^{3/4} 1\,\sin n\pi x\,dx \\
&= 2\Big[-\frac{1}{n\pi}\Big]\Big[\cos n\pi x\Big]_{1/2}^{3/4} \\
&= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{3n\pi}{4}\right].
\end{aligned} \tag{76}$$

Hence the final solution is,

$$\theta = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{3n\pi}{4}\right] \sin n\pi x. \tag{77}$$

The trigonometric terms in the expression for B_n cannot be simplified and therefore we just have to write them as they are. When creating the temperature profiles it is certainly not a hassle for the computer to handle them for evaluation and/or plotting purposes.

The following figure shows the evolution of the temperature with time.

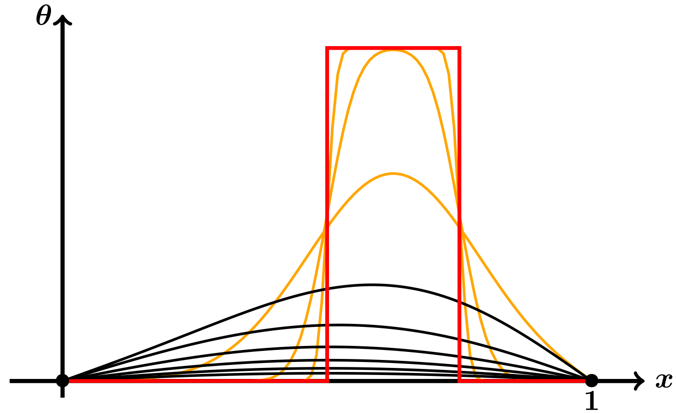


Figure 4.4. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05\pi^2/\alpha$. Orange profiles correspond to $0.0001\pi^2/\alpha$, $0.001\pi^2/\alpha$ and $0.01\pi^2/\alpha$.

At very early times the only change from the initial profile is in the near vicinity of the discontinuities at $x = \frac{1}{2}$ and $\frac{3}{4}$. Thus heat is beginning to diffuse from what we could call the thermal pulse which not only cools the edges of the pulse but also begins to heat the external regions. When $\alpha t = 0.0001\pi^2$, there are distinct thermal boundary layers formed which are centred at $x = \frac{1}{2}$ and $x = \frac{3}{4}$. As time progresses these boundary layers thicken until they merge and only then does the temperature at the centre of the original pulse ($x = \frac{5}{8}$) begin to decrease. Eventually the $n = 1$ component of the solution dominates leaving us with the familiar half sine wave.

Clearly, the temperature at $x = \frac{5}{8}$ remains constant for an interval of time and then it decays to zero, but it is of some interest to think about the evolution of the temperature at a point outside of where the initial pulse is, such as $x = \frac{7}{8}$. When $t = 0$ we have $\theta = 0$ at $x = \frac{7}{8}$. The temperature remains at zero for a short while until the heat diffuses towards that point. Thus the temperature first increases to a maximum and then eventually decays back to zero. I have deliberately not given a figure to show this, for I think that it is a good exercise to glean the rough shape of this evolution from the curves shown in Fig. 4.4.

4.2 Fourier Sine Series Solutions of the wave equation.

We shall do now for the wave equation what we have just done for the Fourier's equation. In Examples 4.5 to 4.8 we will solve the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{78}$$

subject to $y = 0$ on $x = 0, 1$ (i.e. the taut string has no displacement at its end points) and with the initial displacement, $y = f(x)$, and zero velocity at $t = 0$. I shall leave $f(x)$ to be unspecified at present and then we'll consider some specific examples afterwards. Note that the wave equation has a second derivative with respect to t and therefore it needs two initial conditions. The diagram of the configuration may be found in §2.3 and ought to be the initial part of one's analysis.

The separation of variables ansatz is to let

$$y(x, t) = T(t) \sin n\pi x, \tag{79}$$

where these sines have been chosen in order to satisfy the boundary conditions at $x = 0$ and $x = 1$. The substitution into the wave equation gives,

$$T'' \cancel{\sin n\pi x} = -n^2\pi^2c^2T \cancel{\sin n\pi x}$$

$$\implies \qquad T'' = -n^2\pi^2c^2T$$

$$\implies \qquad T'' + n^2\pi^2c^2T = 0 \qquad \text{The simple harmonic motion equation}$$

$$\implies \qquad T = A\cos n\pi ct + B\sin n\pi ct \qquad \text{n.b. we're expecting wavy solutions in time!}$$

$$\implies \qquad y = \underbrace{\Big[A\cos n\pi ct + B\sin n\pi ct\Big]\sin n\pi x.}_{\text{fundamental solution}} \tag{80}$$

An easy way to remember that the solution of $T'' + n^2\pi^2c^2T = 0$ is composed of sines and cosines, rather than exponentials, is to note that we are solving the wave equations, and we expect waves! Now we superpose all the fundamental solutions for $n = 1, 2, 3\cdots$:

$$y = \sum_{n=1}^{\infty}\Big[A_n\cos n\pi ct + B_n\sin n\pi ct\Big]\sin n\pi x. \tag{81}$$

Then we need to apply the initial conditions. It is better to apply the zero-velocity condition first: if $\partial y/\partial t = 0$ at $t = 0$ then $B_n = 0$ for all values of n. The B_n coefficients multiply the sine terms all of which have a nonzero derivative at $t = 0$, and therefore they are eliminated. On the other hand the A_n coefficients, which multiply cosines already have zero gradients at $t = 0$. Therefore Eq. (81) reduces to

$$y = \sum_{n=1}^{\infty} A_n\cos n\pi ct\sin n\pi x. \tag{82}$$

The final initial condition is that $y = f(x)$ at $t = 0$, and therefore,

$$f(x) = \sum_{n=1}^{\infty} A_n\sin n\pi x, \tag{83}$$

which is a Fourier Sine Series.

Note: By convention we usually use B_n as the coefficient for sines, so we need to be careful at this point and resolute in affirming that this is indeed a Fourier Sine Series. So the Fourier Coefficient is given by,

$$A_n = 2\int_0^1 f(x)\,\sin n\pi x\,dx. \tag{84}$$

Now we shall consider five different initial displacements in five separate examples. Each will display a different facet of the different types of solutions that can arise. When a PDE is solved in an exam or from a problem sheet question, the final result is a somewhat boring-looking summation. However, the plotting of that solution provides a lot of physical intuition about it, and this is why I will be providing many graphs of the final solutions. Now we'll get on to the examples.

Example 4.5. A very simple initial condition is $y = \sin \pi x$ and we shall couple this with a zero velocity at $t = 0$.

For this we will substitute the initial condition into Eq. (83) to obtain,

$$\sin \pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x. \tag{85}$$

While the values of A_n may be found by applying Eq. (84) it is easier and much quicker simply to compare like-terms each side. This gives us,

$$A_1 = 1 \quad \text{and} \quad 0 = A_2 = A_3 = A_4 = \cdots . \tag{86}$$

Hence the final solution is,

$$y = \cos \pi ct \, \sin \pi x. \tag{87}$$

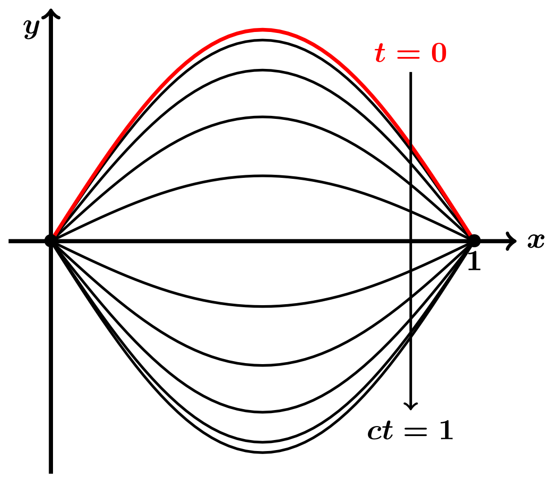


Figure 4.5. Depicting half a period of a solution of the wave equation. The initial condition is displayed in red. Successive profiles are at a time-interval of $0.1/c$. The full period of oscillation is $2/c$.

So clearly this taut string continues to oscillate without change of shape and with an amplitude that varies sinusoidally in time. Only the first half of a period is shown above, and the second half has the string essentially retracing its steps, so to speak, until it returns to the inital state after one period.

Example 4.6. In this case will choose to employ a parabolic profile as the initial displacement $y = x(1-x)$ at $t = 0$. The Fourier sine coefficient is given by the by-now standard formula:

$$A_n = 2\int_0^1 x(1-x)\,\sin n\pi x\, dx = \begin{cases} \dfrac{8}{n^3\pi^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{88}$$

The final solution is,

$$y = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} \cos n\pi ct \sin n\pi x, \tag{89}$$

and the following figure shows half a period of the subsequent motion.

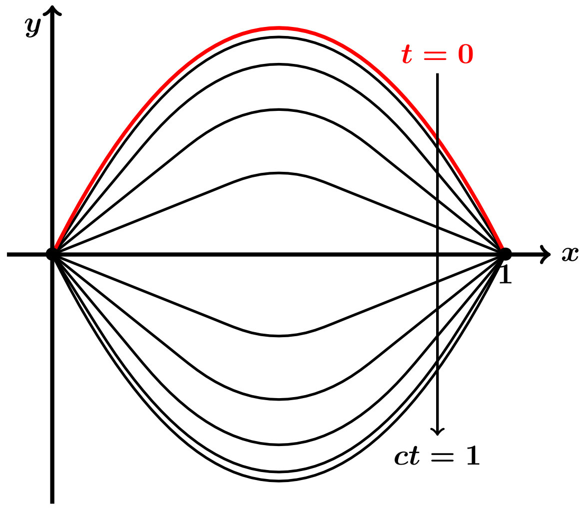


Figure 4.6. Depicting a solution of the wave equation pver half a period. The initial condition is displayed in red. Successive profiles are at a time-interval of $0.1/c$.

A brief glance suggests that very little has changed between Fig. 4.5 and Fig. 4.6, but the shape of the profile looks as though it is approaching a piecewise linear form as it nears the horizontal axis. This suggests that the velocity profile will be piecewise-linear. I certainly wouldn't be asking why this is the case as part of an exam question because you can't be asked to plot the graphs, but right now I am not constrained in that way! However, I'll just give an outline of the proof that the velocity of the string is a piecewise linear function of x when $t = 1/(2c)$, i.e. after a quarter of a period. If one finds the time-derivative of y from Eq. (89), this yields a solution where the terms have $n^2\pi^2$ in the denominator, as opposed to $n^3\pi^3$ for y itself. The substitution of $t = 1/2c$ into this gives a function which is a multiple of that given in Eq. (71) when $t = 0$, which is indeed a piecewise linear function. If you're interested in checking this out then do so, otherwise you may ignore this paragraph with impunity.

Example 4.7. Now we shall consider the piecewise-linear initial condition,

$$y = f(x) = \begin{cases} 2x & (0 \le x \le \frac{1}{2}) \\ \\ 2 - 2x & (\frac{1}{2} \le x \le 1) \end{cases} \quad \text{at } t = 0, \tag{90}$$

which defines $f(x)$, The final solution is given by Eq. (82) where A_n is given by,

$$A_n = 2\int_0^1 f(x)\,\sin n\pi x\,dx = \frac{8\sin(\frac{1}{2}n\pi)}{n^2\pi^2.} \tag{91}$$

We have already covered an almost identical integration in Example 4.3. The final solution is,

$$y = \sum_{n=1}^{\infty} \frac{8\sin(\frac{1}{2}n\pi)}{n^2\pi^2}\cos n\pi ct\,\sin n\pi x, \tag{92}$$

and just over a quarter of a period of the subsequent motion is given in the following figure.

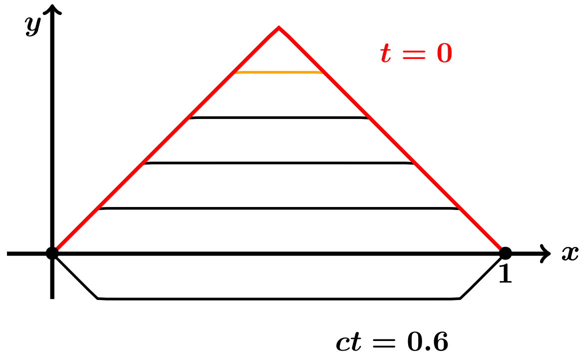


Figure 4.7. Depicting a solution of the wave equation. The initial condition is displayed in red. The orange profile corresponds to $t = 0.1/c$. Successive profiles are at a time-interval of $0.1/c$.

First, it is essential to say that a piecewise-linear initial profile is impossible to set up in practice and therefore this solution will only ever be an approximation to reality. The thinner the taut string, the closer one can get to this idealised situation but even then that part of the string which is within a string thickness of the corner will be placed under enormous bending stresses locally — the following is far from being the best analogy but think of trying to bend a 2cm thick wooden plank where all of the bend takes place within say 4cm of the centre of the bend while the rest of the plank is straight. So this solution has a mathematical interest but only an approximate relationship to reality!

That said, the subsequent motion is quite remarkable and perhaps unpredictable? As soon as the string is released to move, there is a central portion which becomes horizontal while the rest of the string has remained immobile — see the orange profile. As time progresses the horizontal portion expands until, at one quarter of the period ($t = 1/(2c)$) the whole string is horizontal. The remaining evolution may now be predicted easily.

Example 4.8. The initial condition for this example is

$$y = f(x) = \begin{cases} x/a & (0 \le x \le a) \\ \\ (1-x)/(1-a) & (a \le x \le 1) \end{cases} \quad \text{at } t = 0. \tag{93}$$

This is again a piecewise-linear profile (see the red curve in Fig. 4.8, below) but where the location of the 'join' is at $x = a$. This is an approximation to the profile which is created when plucking a violin string. In the classical musical world this act of plucking is called pizzicato.

The Fourier coefficient is given by

$$\begin{aligned} A_n &= 2\int_0^1 f(x)\sin n\pi x\, dx \\ &= 2\int_0^a \frac{x}{a}\sin n\pi x\, dx + 2\int_a^1 \frac{(1-x)}{(1-a)}\sin n\pi x\, dx \\ &= \frac{2\sin(n\pi a)}{n^2\pi^2(a-a^2)}. \end{aligned} \tag{94}$$

Therefore the final solution is

$$y = \sum_{n=1}^{\infty} \frac{2\sin(n\pi a)}{n^2\pi^2(a - a^2)} \cos n\pi ct \, \sin n\pi x, \tag{95}$$

and it is shown below.

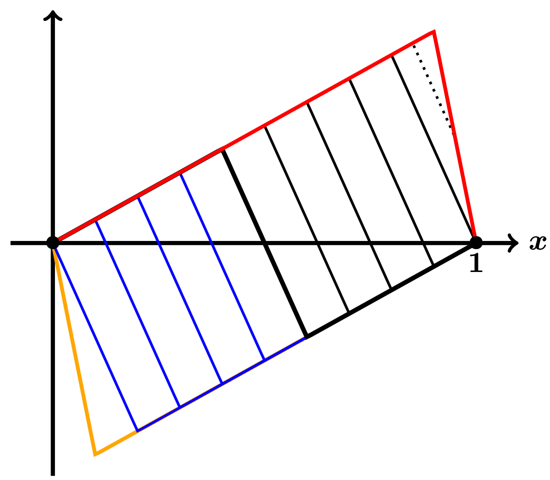


Figure 4.8. Depicting the solution of the wave equation given in Eq. (93) with $a = 0.9$ over half a period. The initial condition is displayed in red while the solution half a period later at $t = 1/c$ is displayed in orange. Successive profiles are at a time-interval of $0.1/c$ although the dotted profile is at $t = 0.05/c$.

As mentioned, the red 'curve' is a typical pizzicato profile for a violin and, more generally for a stringed instument. As soon as the string is released the string evolves immediately into a piecewise linear profile consisting of three straight lines as depicted by the dotted line and the remaining part of the red profile which remains immobile.

The first continuous black line corresponds to $t = 0.1c$ marks the transition to a different piecewise-linear profile consisting of three straight lines. Now the shape of the string has two right angled corners, with the middle section travelling to the left as time progresses. Eventually the orange curve is reached after half a period ($t = 1/c$).

Example 4.9. The final initial condition which we'll use is,

$$y = f(x) \begin{cases} 0 & (0 \le x \le a) \\ 4\dfrac{(x-a)(b-x)}{(b-a)^2} & (a \le x \le b) \quad \text{at } t = 0. \\ 0 & (b \le x \le 1) \end{cases} \tag{96}$$

In this case we shall use $a = 0.6$ and $b = 0.8$ in order to determine the evolution of an isolated pulse in the middle of the string. This initial condition looks like the following,

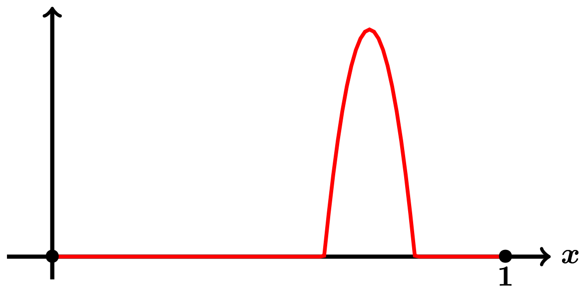


Figure 4.9a. Depicting the initial condition given in Eq. (96). This pulse takes a parabolic shape between $x = 0.6$ and $x = 0.8$ with a zero deflection elsewhere.

The Fourier coefficients for this curve are given by,

$$A_n = 2\int_0^1 f(x)\,\sin n\pi x\,dx = 16\frac{\cos n\pi a - \cos n\pi b}{(b-a)^2 n^3\pi^3} - 8\frac{\sin n\pi b + \sin n\pi a}{(b-a)n^2\pi^2}, \tag{97}$$

and therefore the final solution is,

$$y = \sum_{n=1}^{\infty}\left[16\frac{\cos n\pi a - \cos n\pi b}{(b-a)^2 n^3\pi^3} - 8\frac{\sin n\pi b + \sin n\pi a}{(b-a)n^2\pi^2}\right]\cos n\pi ct\sin n\pi x. \tag{98}$$

Again, such a complicated solution may be evaluated and plotted easily using Matlab.

The resulting evolution of the displacement needs a few Figures to describe it well. In this first one, Figure 4.9b, we see an interesting phenomenon whereby the initial pulse splits into two identical pulses each with half the height of the original but travelling in opposite directions with identical speeds.

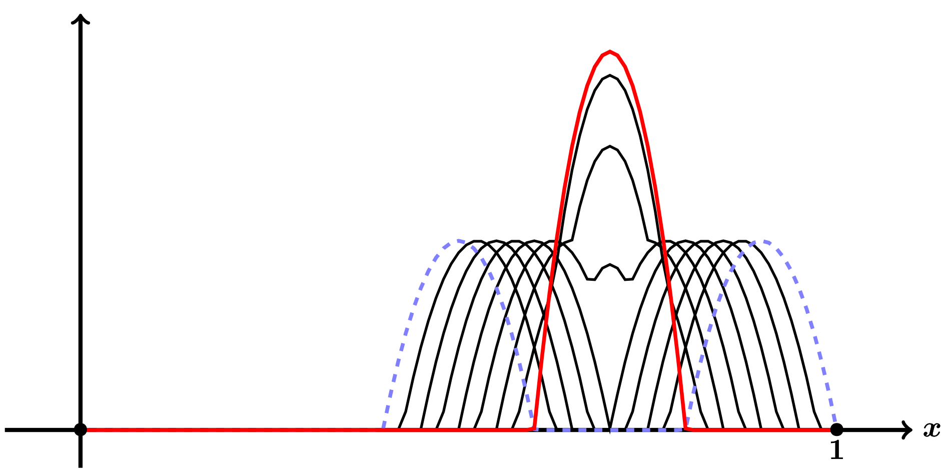
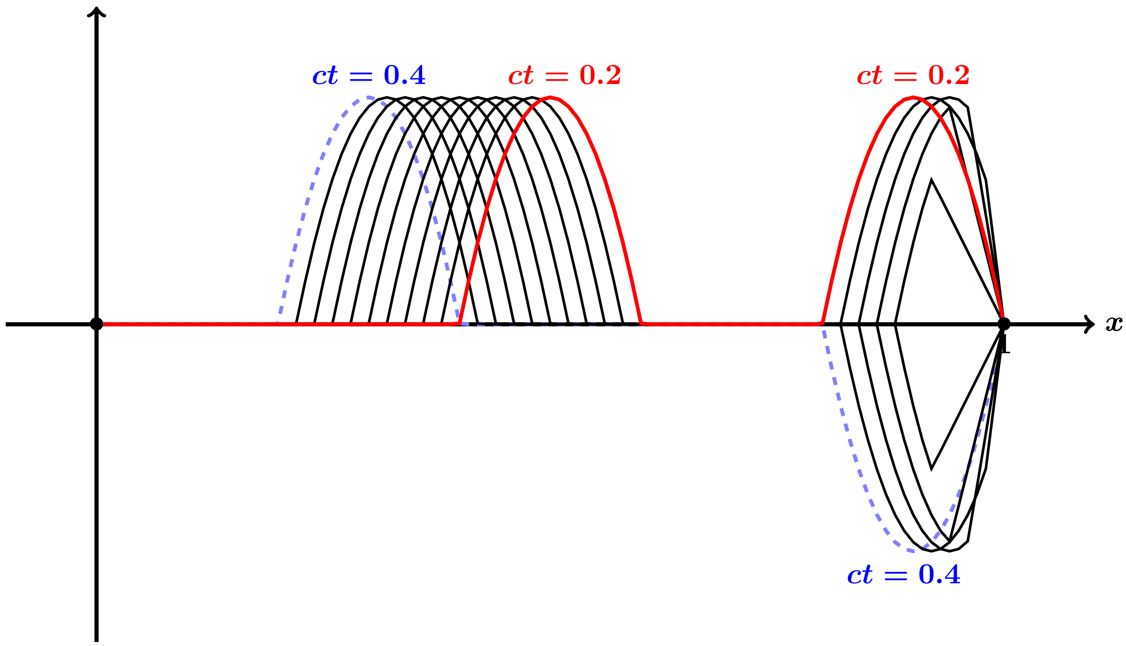


Figure 4.9b. Depicting a solution of the wave equation. The initial condition at $t = 0$ is displayed in red, while the dashed blue profile corresponds to $ct = 0.2$. Successive profiles are at a time-interval of $0.025/c$.

In this first phase of movement the splitting of the initial pulse is complete by the time $ct = 0.1$ and thereafter we have two pulses moving one to the left an one to the right. The blue curve represents $ct = 0.2$ which is when the right hand pulse encounters the right hand boundary.



Figute 4.9c. Depicting a solution of the wave equation. The profile at $ct = 0.2$ is in red, while the one at $ct = 0.4$ is represented by a blue dashed line. Successive profiles are at a time-interval of $0.02/c$.

In this interval of time from $ct = 0.2$ to $ct = 0.4$ the left hand pulse continues to travel to the left with speed, c, and it will collide with the left hand boundary at $ct = 0.6$. Meanwhile the rightward-moving right hand pulse undergoes a full reflection where the pulse at $ct = 0.2$ (red) is transformed into an inverted form at $ct = 0.4$ (blue) which is now travelling leftwards. At precisely $ct = 0.3$ this pulse disappears completely. Eventually the left hand pulse will travel further to the left, be reflected from the $x = 0$ boundary, been inverted and then starts to travel right with a negative amplitude. When $ct = 1$ this pulse merges completely with the other pulse at $x = 0.3$ to form a single pulse with a negative amplitude. At this point half a period has been completed.

Example 4.10. A solution of Laplace's equation.
We shall solve Laplace's equation subject to $\theta = 0$ on both $x = 0$ and $x = 1$ and with $\theta = x$ on $y = 0$. For the sake of brevity we'll dispense with a diagram here (althoughit is always advisable in an exam context), but Figure 2.4 with $f(x) = x$ describes the configuration and the boundary conditions perfectly. Laplaces' equations is,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0. \tag{99}$$

Again for the sake of brevity we'll omit the analysis which follows Figure 2.4 and which eventually arrives at Eq. (24) quoted here:

$$\theta = \sum_{n=1}^{\infty} B_n e^{-n\pi y} \sin n\pi x. \tag{100}$$

So this expression satisfies Laplace's equation, the boundary conditions at $x = 0$ and $x = 1$ and the physical requirement that $\theta \to 0$ as $y \to 0$. Now we need to apply the final boundary condition that $\theta = x$ when $y = 0$. Hence we obtain the Fourier Sine series,

$$x = \sum_{n=1}^{\infty} B_n \sin n\pi x, \tag{101}$$

where the Fourier coefficients are given by applying Eq. (46) with $d = 1$. We obtain, XXXX —To be updated/added to for 2022/23—

5 Fourier Cosine Series solutions of Fourier's equation

Once more recall that the Fourier's equation is,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}. \tag{102}$$

The difference between this section and the last is that we shall be employing Neumann conditions on both boundaries. We'll proceed in the same way as in the last section, namely to determine a general solution where everything except for the Fourier Series has been found. Then we'll run through a few cases.

Example 5.1. Solve Fourier's equation where $\partial\theta/\partial x = 0$ on $x = 0$ and $x = 1$ where the initial condition is, $\theta = x$ at $t = 0$.

Given the boundary conditions we will let $\theta = T(t)\cos(n\pi x)$, where n is either zero or else a positive integer. In a Fourier Cosine Series (unlike a Fourier Sine Series) we do need to include the $n = 0$ term, a straight line, because it satisfies the boundary conditions; see Fig. 3.3. Hence,

$$\begin{aligned}\frac{dT}{dt}\cancel{\cos(n\pi x)} &= -\alpha n^2\pi^2 T\cancel{\cos(n\pi x)} \\ \implies \quad \frac{dT}{dt} &= -\alpha n^2\pi^2 T \\ \implies \quad T &= Ae^{-\alpha n^2\pi^2 t} \quad \text{when } n \neq 0\end{aligned} \tag{103}$$

and

$$\frac{dT}{dt} = 0 \qquad \implies \qquad T = \tfrac{1}{2}A_0 \quad \text{when } n = 0. \tag{104}$$

Note that a ½ has been used here as the coeffícent of A_0 because it fits in with the standard form of the Fourier Cosine Series.

On reconstructing θ, and superposing all the valid solutions, we have

$$\theta = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha n^2\pi^2 t}\cos(n\pi x). \tag{105}$$

This will be used as the 'starting point' for later examples.

Now we apply the Initial Condition that $\theta = x$ when $t = 0$ and therefore Eq. (105) yields,

$$x = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x). \tag{106}$$

This is a Fourier Cosine Series and the coefficients are given by,

$$\begin{aligned}A_n &= 2\int_0^1 x\cos(n\pi x)\,dx \qquad = \qquad 2\Big[\underbrace{\Big(x\Big)}_{D_0}\underbrace{\Big(\frac{\sin n\pi x}{n\pi}\Big)}_{I_1} - \underbrace{\Big(1\Big)}_{D_1}\underbrace{\Big(-\frac{\cos n\pi x}{n^2\pi^2}\Big)}_{I_2}\Big]_0^1 \\ &= \frac{2}{n^2\pi^2}\Big[\cos n\pi x\Big]_0^1 \\ &= \frac{2}{n^2\pi^2}\Big((-1)^n - 1\Big) \\ &= \begin{cases} -\dfrac{4}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even}\end{cases}\end{aligned} \tag{107}$$

We also find that,

$$A_0 = 2\int_0^1 x\,dx = 1. \tag{108}$$

Hence the final solution is,

$$\theta = \tfrac{1}{2} - \sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{4}{n^2\pi^2} e^{-\alpha n^2\pi^2 t}\cos n\pi x. \tag{109}$$

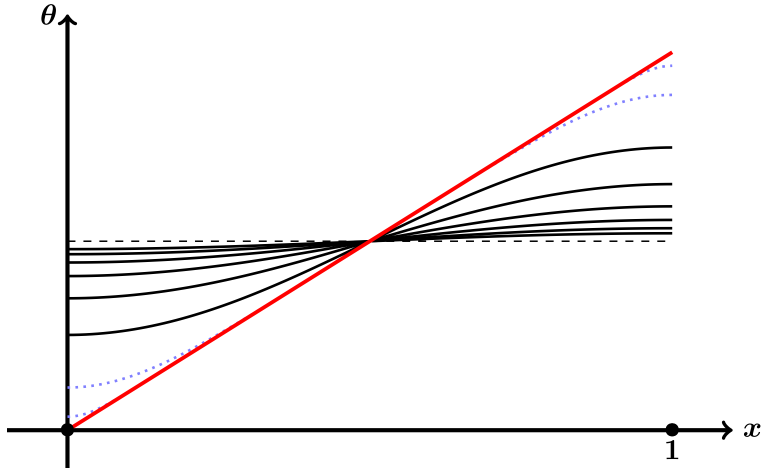


Figure 5.1. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05/\alpha\pi^2$. The two dotted blue profiles correspond to $t = 0.001/\alpha\pi^2$ and $t = 0.01/\alpha\pi^2$.

In this figure we see that the initial temperature profile doesn't satisfy the Neumann conditions at $x = 0$ and $x = 1$, and therefore the very first stage of its evolution is centred on fixing that mismatch. In the meantime, the rest of the profile is not affected because $\theta_{xx} = 0$ in the interior (Proof: $\theta = x \Rightarrow \theta_{xx} = 0 \Rightarrow \theta_t = 0$). Eventually the modifications to the profile near the boundaries diffuse inwards and the temperature relaxes to a uniform constant state which is the mean temperature of the initial profile ($\theta_x = 0$ on the boundaries means that there is no heat lost from the system), and this is indicated by the dashed line in the figure. Crudely, one may think of the imperfect analogy of having these curves represent the height of a very viscous fluid in a container — the initial linear profile relaxes down to a uniform level as time progresses.

Example 5.2. We will now use the initial condition,

$$\theta = f(x) = \begin{cases} 1 & (0 < x < \tfrac{1}{2}) \\ 0 & (\tfrac{1}{2} < x < 1) \end{cases} \qquad \text{at } t = 0, \tag{110}$$

which defines $f(x)$.

The full analysis would involve the separation of variables analysis which leads to Eq. (105). We omit it here but it would be expected in an exam. Therefore the next task is to determine A_n and A_0.

Given that $f(x)$ is defined in a piecewise manner we need a little care:

$$\begin{aligned}
A_n &= 2\int_0^2 f(x)\cos n\pi x\,dx \\
&= 2\int_0^{1/2} 1\ \cos n\pi x\,dx + \cancel{2\int_{1/2}^{1} 0\ \cos n\pi x\,dx} \\
&= 2\int_0^{1/2} 1\ \cos n\pi x\,dx \\
&= \frac{2\sin\frac{n\pi}{2}}{n\pi}.
\end{aligned} \tag{111}$$

We also find that $A_0 = 1$. Hence the final solution is,

$$\theta = \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{2\sin\frac{n\pi}{2}}{n\pi} e^{-\alpha n^2\pi^2 t}\cos n\pi x. \tag{112}$$

The evolution of the initial profile looks like this:

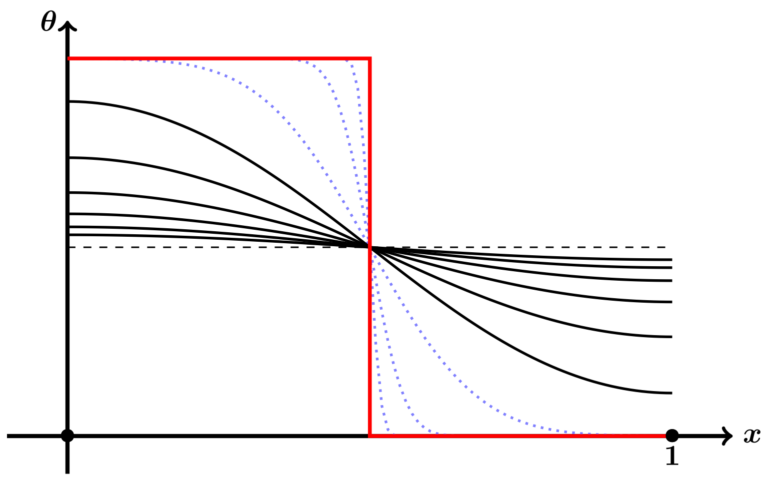


Figure 5.2. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05/\alpha\pi^2$. The three dotted blue profiles correspond to $t = 0.0001/\alpha\pi^2$, $t = 0.001/\alpha\pi^2$ and $t = 0.01/\alpha\pi^2$.

So the initial condition consists of a uniformly hot region and a uniformly cold region. The immediate change to this profile as time progresses is that the hot region begins to warm up the cold region and vice versa. The blue curves show this process. Essentially we have a very thin internal thermal boundary layer at early times which gradually thickens until it reaches the boundaries of the domain at $x = 0$ and $x = 1$. Thereafter the temperature profile evolves towards a completely uniform state since heat cannot escape from the domain. At very late times the solution is well-approximated using the first partial sum in Eq. (112):

$$\theta \simeq \tfrac{1}{2} + \frac{2}{n\pi} e^{-\alpha\pi^2 t}\cos\pi x. \tag{113}$$

Finally, we note that the term, $\sin(n\pi/2)$, in Eq. (112) is zero when n is even. We may therefore rewrite the solution in the form,

$$\theta = \tfrac{1}{2} + \sum_{m=0}^{\infty} \frac{2(-1)^m}{(2m+1)\pi} e^{-\alpha(2m+1)^2\pi^2 t} \cos(2m+1)\pi x. \tag{114}$$

Example 5.3. We'll consider the following initial condition:

$$\theta = 4x(1-x) \quad \text{at } t = 0. \tag{115}$$

Omitting the details of the analysis we obtain the final solution,

$$\theta = \tfrac{2}{3} - \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{16}{n^2\pi^2} e^{-\alpha n^2\pi^2 t} \cos n\pi x. \tag{116}$$

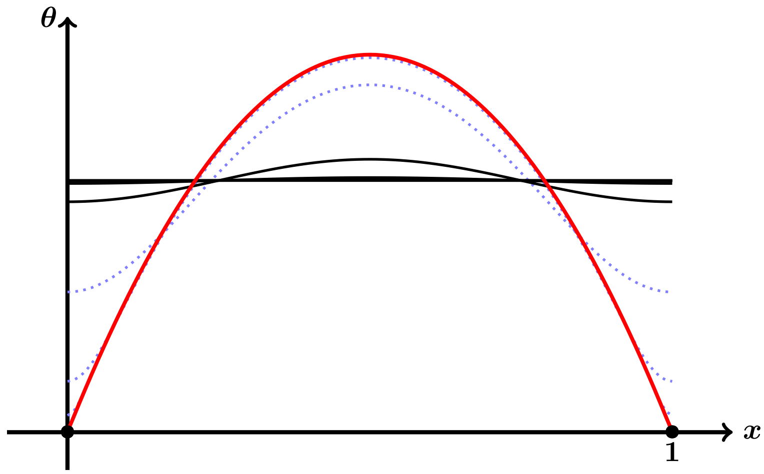


Figure 5.3. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.05/\alpha\pi^2$. The three dotted blue profiles correspond to $t = 0.0001/\alpha\pi^2$, $t = 0.001/\alpha\pi^2$ and $t = 0.01/\alpha\pi^2$.

Given the previous two Examples this one provides no surprises. The initial heat distribution diffuses in such a way that a uniform profile is obtained as $t \to \infty$. But the one noteworthy difference between this solution and very many others that involve a Fourier Series of some sort is that the summation is over even values of n. But why is this? If one were to consider the symmetries of the four cosine profiles given in Fig. 3.3 then it will be seen that some are even about the mid-point and some are odd. It is generally the case that a function which is odd can only be represented by odd functions in its Fourier Series and, likewise, an even function can only be represented using even functions. In this present example the initial profile is even about $x = 1/2$ and the Fourier Series consists solely of functions which are also even about $x = 1/2$; see the symmetries of the cosines displayed in Fig. 3.3.

In view of this we may let $n = 2m$ which means that Eq. (116) may be replaced by.

$$\theta = \tfrac{2}{3} - \sum_{m=1}^{\infty} \frac{4}{m^2\pi^2} e^{-4\alpha m^2\pi^2 t} \cos 2m\pi x. \tag{117}$$

6 Quarter-range Fourier Series solutions for Fourier's equation

Yet again we recall that Fourier's equation is,

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}, \tag{118}$$

and we shall solve this subject to the boundary conditions,

$$\theta = 0 \text{ at } x = 0 \qquad \text{and} \qquad \frac{\partial\theta}{\partial x} = 0 \text{ at } x = 1. \tag{119}$$

Two examples with different initial conditions will be considered.

Given the boundary conditions we will use

$$\theta = T(t)\sin(n\pi x/2) \tag{120}$$

as the separation of variables ansatz, where n is a positive odd integer. Substitution into Fourier's equation yields,

$$\frac{dT}{dt} = -(\alpha n^2\pi^2/4)\,T \qquad \Longrightarrow \qquad T = Be^{-\alpha n^2\pi^2 t/4}. \tag{121}$$

On reconstructing θ, and superposing all the valid solutions, we have

$$\theta = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} B_n e^{-\alpha n^2\pi^2 t/4}\sin(n\pi x/2). \tag{122}$$

The summation over odd values of n is because the expression in Eq. (120) satisfies the given boundary conditions when n is odd but not otherwise. This completes the separation of variables analysis, and now we turn to the two different initial conditions.

Example 6.1. We shall consider the evolution of the initial profile,

$$\theta = x \text{ at } t = 0. \tag{123}$$

Substitution of this into Eq. (122) yields,

$$x = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} B_n e^{-\alpha n^2\pi^2 t/4}\sin(n\pi x/2). \tag{124}$$

The Fourier coefficients are given by,

$$B_n = 2\int_0^1 x\,\sin(n\pi x/2)\,dx = \frac{8\sin(n\pi/2)}{n^2\pi^2}, \tag{125}$$

after some integration by parts. Substitution of this back into Eq. (120) followed by superposition for odd values of n yields the final solution,

$$\theta = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{8\sin(n\pi/2)}{n^2\pi^2} e^{-\alpha n^2\pi^2 t/4}\sin(n\pi x/2). \tag{126}$$

This solution looks like,

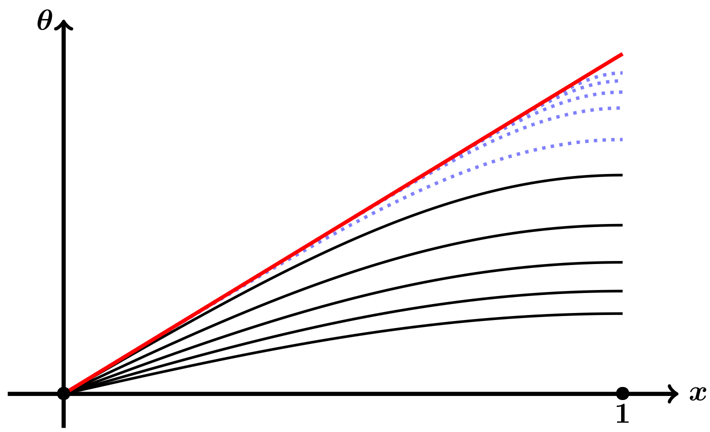


Figure 6.1. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.1/\alpha\pi^2$. The dotted blue profiles correspond to $\alpha\pi^2 t = 0.0025$, 0.005, 0.01, 0.02 and 0.05.

The evolution of θ with time is quite straightforward. Given that the $x = 1$ boundary is insulated no heat is lost from it even though the temperature at $x = 1$ clearly decreases with time. But heat is lost from the $x = 0$ boundary because the fact that $\partial\theta/\partial x = 1$ at early times means that there is a continuous loss of heat content in the full system.

Example 6.2. Now we shall apply the Initial Condition that $\theta = x - x^2$ when $t = 0$. This gives,

$$x - x^2 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B_n \sin(n\pi x/2). \tag{127}$$

The Fourier coefficients are given by,

$$\begin{aligned} B_n &= 2\int_0^1 (x - x^2)\sin(n\pi x/2)\,dx \\ &= 2\Big[\underbrace{\Big(x - x^2\Big)}_{D_0}\underbrace{\Big(-\frac{2\cos n\pi x/2}{n\pi}\Big)}_{I_1} - \underbrace{\Big(1 - 2x\Big)}_{D_1}\underbrace{\Big(-\frac{4\sin n\pi x/2}{n^2\pi^2}\Big)}_{I_2} + \underbrace{\Big(-2\Big)}_{D_2}\underbrace{\Big(\frac{8\cos n\pi x/2}{n^3\pi^3}\Big)}_{I_3}\Big]_0^1 \\ &= \frac{32}{n^3\pi^3} - \frac{8\sin(n\pi/2)}{n^2\pi^2} \end{aligned} \tag{128}$$

Therefore the final solution is,

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\frac{32}{n^3\pi^3} - \frac{8\sin(n\pi/2)}{n^2\pi^2}\right) e^{-\alpha n^2\pi^2 t/4} \sin n\pi x/2. \tag{129}$$

Yes, the Fourier coefficient is complicated but that is of no consequence.

The solution looks like the following:

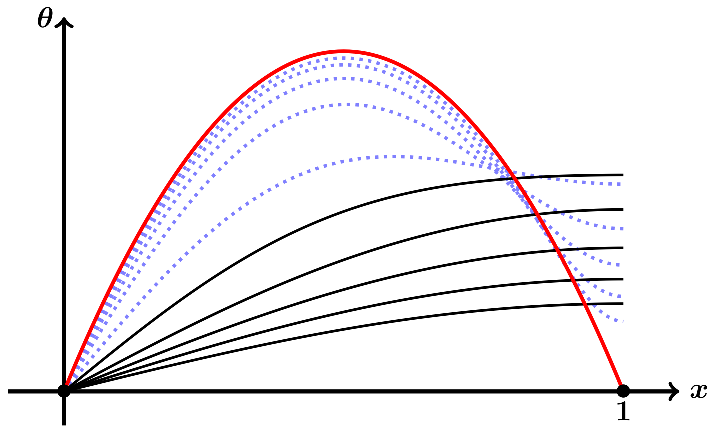


Figure 6.2. Depicting the above solution of Fourier's equation. The initial condition is displayed in red. Successive black profiles are at a time-interval of $0.1/\alpha\pi^2$. The dotted blue profiles correspond to $\alpha\pi^2 t = 0.0025$, 0.005, 0.01, 0.02 and 0.05.

As with Fig. 6.1, heat continues to be lost from the $x = 0$ boundary and therefore the ultimate fate is $\theta = 0$. But heat does spread to the right initially, which is what causes the temperature at the $x = 1$ boundary to rise at first. Eventually the heat loss on the left retards the rise in temperature at $x = 1$ until it too decreases towards zero.

7 Solutions of Laplace's equation in finite domains.

Thus far we done little to solve Laplace's equation apart from some separation of variables analyses in §3. But there are very strong similarities between the solutions of the three main PDEs which form the focus of our attention, as we'll see now.

We may solve **Fourier's equation** with $\theta = x(1-x)$ at $t = 0$ and with $\theta = 0$ at $x = 0$ and $x = 1$. This solution may be found in Eq. (64):

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} e^{-\alpha n^2\pi^2 t} \sin n\pi x. \tag{130}$$

We may also solve the **wave equation** with the same boundary conditions and with the displacement equal to $x(1-x)$ at $t = 0$ and with a zero velocity. Its solution may be found from Eq. (89):

$$y = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} \cos n\pi ct \sin n\pi x. \tag{131}$$

And finally we may solve **Laplace's equation** with zero values of θ on $x = 0$, $x = 1$ and as $y \to \infty$, and with $\theta = x(1-x)$ on $y = 0$. This solution is,

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3} e^{-n\pi y} \sin n\pi x. \tag{132}$$

Of course, there is a clash of notation above because y is used as a displacement for the wave equation and as a coordinate for Laplace's equation; hopefully that is not confusing. But the message here is that the separation of variables part of the solution is almost completely independent of the determination of the Fourier coefficients.

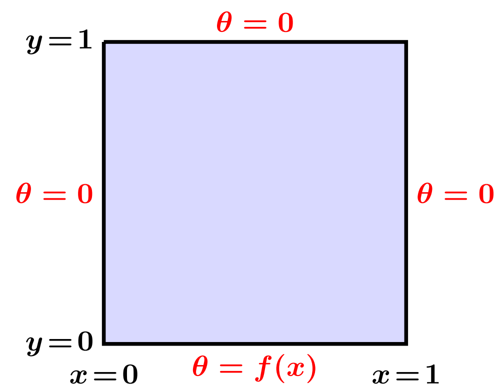
In this section we will be considering square domains, although solutions for rectangular domains require a small tweak....

As a reminder, Laplace's equation is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0. \tag{133}$$

We shall solve this equation in a square domain ($0 \le x \le 1$ and $0 \le y \le 1$) with a variety of boundary conditions. It will prove easiest to indicate these boundary conditions diagrammatically, rather than to use too many words.

Example 7.1. The domain is given by,



We will employ separation of variables again but this will require us to decide on which direction uses sines. Here, we have $\theta = 0$ boundary conditions at $x = 0$ and $x = 1$ and therefore we will use $\sin n\pi x$.

At $y = 0$ we have $\theta = f(x)$ which will mean that a Fourier Series will eventually need to be used.

As earlier, we will begin with the separation of variables ansatz. Let

$$\theta = Y(y)\sin n\pi x, \tag{134}$$

where n may take positive integer values. This yields the following equation for Y:

$$Y'' - n^2\pi^2 Y = 0. \tag{135}$$

The general solution of this ODE is,

$$Y = Ae^{n\pi y} + Be^{-n\pi y}, \tag{136}$$

and therefore,

$$\theta = \Big[Ae^{n\pi y} + Be^{-n\pi y}\Big]\sin n\pi x. \tag{137}$$

When we got to this point in §2.2 and Eq. (22), we invoked the fact that θ must decay as y becomes large because the heat which is supplied at $y = 0$ where $\theta = f(x)$ is lost to the vertical boundaries, and hence we had to remove the growing exponential term because it is unphysical. By contrast, this Example occupies only a finite domain and therefore we must retain both of the exponential terms.

Now we need to superpose all of the solutions represented by Eq. (137):

$$\theta = \sum_{n=1}^{\infty}\Big[A_n e^{n\pi y} + B_n e^{-n\pi y}\Big]\sin n\pi x. \tag{138}$$

Now we need to apply the boundary conditions in the y-direction.

$$y = 0: \qquad f(x) = \sum_{n=1}^{\infty} \Big(A_n + B_n \Big) \sin n\pi x, \tag{139}$$

and

$$y = 1: \qquad 0 = \sum_{n=1}^{\infty} \Big(A_n e^{n\pi} + B_n e^{-n\pi} \Big) \sin n\pi x. \tag{140}$$

If we were to let

$$f(x) = \sum_{n=1}^{\infty} C_n \sin n\pi x \qquad \text{where} \qquad C_n = 2 \int_0^1 f(x) \sin n\pi x \, dx, \tag{141}$$

then we may equate coefficients of like-terms in each of Eqs. (139) and (140):

$$A_n + B_n = C_n \qquad \text{and} \qquad A_n e^{n\pi} + B_n e^{-n\pi} = 0. \tag{142}$$

So we have a pair of simultaneous equations for A_n and B_n where C_n is known. The solution is,

$$A_n = -\frac{C_n e^{-2n\pi}}{1 - e^{-2n\pi}} \qquad \text{and} \qquad B_n = \frac{C_n}{1 - e^{-2n\pi}}. \tag{143}$$

Therefore,

$$\theta = \sum_{n=1}^{\infty} \Big(A_n e^{n\pi y} + B_n e^{-n\pi y} \Big) \sin n\pi x \tag{144}$$

and hence

$$\theta = \sum_{n=1}^{\infty} C_n \left[\frac{e^{-n\pi y} - e^{n\pi y - 2n\pi}}{1 - e^{-2n\pi}} \right] \sin n\pi x. \tag{145}$$

Equation (145) provides a perfectly adequate solution for an exam context, but this may be made more compact by a sneaky manoeuvre. If we multiply both the numerator and the denominator by $e^{n\pi}$ then,

$$\begin{aligned} \theta &= \sum_{n=1}^{\infty} C_n \left[\frac{e^{n\pi(1-y)} - e^{-n\pi(1-y)}}{e^{n\pi} - e^{-n\pi}} \right] \sin n\pi x \\ &= \sum_{n=1}^{\infty} C_n \left[\frac{\sinh n\pi(1-y)}{\sinh n\pi} \right] \sin n\pi x. \end{aligned} \tag{146}$$

Now we shall choose a simple expression for $f(x)$. If $f(x) = 1$ then $C_n = \dfrac{4}{n\pi}$ for odd values of n and is other wise zero, and so Eq. (146) becomes

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi(1-y)}{\sinh n\pi} \right] \sin n\pi x. \tag{147}$$

This solution is depicted below.

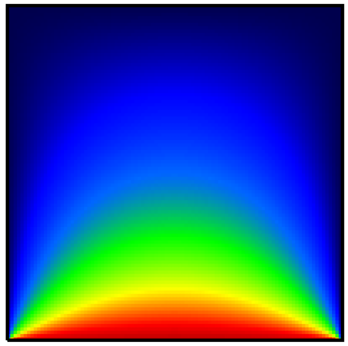


Figure 7.1. Coloured contours for the solution given in Eq. (147). Red indicates hot, as one might expect, and blue represents cold. The lower boundary is held at $\theta = 1$ while the other three boundaries are held at $\theta = 0$.

Example 7.2. Given that Example 7.1 has the solution,

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi(1-y)}{\sinh n\pi} \right] \sin n\pi x,$$

then what does

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi y}{\sinh n\pi} \right] \sin n\pi x \tag{148}$$

represent?

As questions go, this is a little unusual but we can deal with it in the following way.

Note 1. The presence of the $\sin n\pi x$ means that the function is zero on $x = 0$ and $x = 1$.

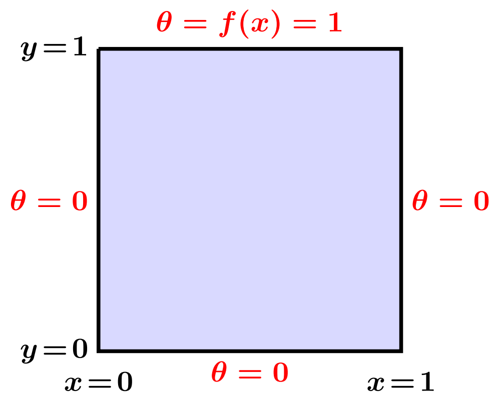
Note 2. Just as $\sinh n\pi(1-y)$ may be written in terms of $e^{n\pi y}$ and $e^{-n\pi y}$, so may $\sinh n\pi y$.

Note 3. The first two notes together tell us that the solution must satisfy Laplace's equation.

Note 4. When $y = 0$ then $\theta = 0$.

Note 5. When $y = 1$ then $\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (4/n\pi) \sin n\pi x$, but this is the FSS for $f(x) = 1$.

So the conclusion is that Eq. (148) satisfies Laplace's equation subject to the following boundary conditions:



This solution (148) looks like the following.

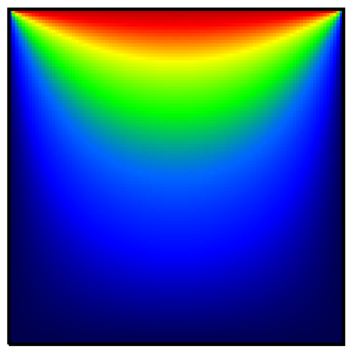


Figure 7.2. Coloured contours for the solution given in Eq. (148) where the upper surface is hot while all the rest are cold.

Example 7.3. This mathematical sleight of hand can be pushed a little further. If we wished to determine the solution for the case where the right hand boundary is heated and the other three held at $\theta = 0$, then all we have to do is to swap all the instances of x and y in Eq. (148). This solution is,

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi x}{\sinh n\pi} \right] \sin n\pi y. \tag{149}$$

A brief check first — clearly $\theta = 0$ when $y = 0$, $y = 1$ and $x = 0$. And when $x = 1$ we regain the Fourier Sine Series for $f(y) = 1$. It will therefore be no surprise that the following figure may be generated from Eq. (149).

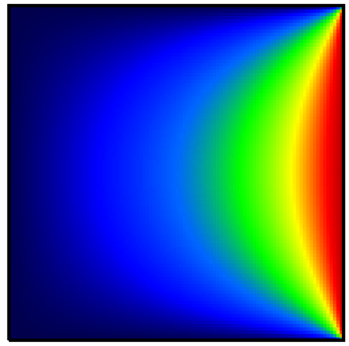


Figure 7.3. Coloured contours for the solution given in Eq. (149) where the right-hand surface is hot while all the rest are cold.

Example 7.4. A final one. If we wish to solve Laplace's equation where $\theta = 1$ on both the upper surface ($y = 1$) and the right-hand surface ($x = 1$) then all we have to do is to add (i.e. superpose!) the solutions given in Eqs. (148) and (149). This is,

$$\theta = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi y}{\sinh n\pi} \right] \sin n\pi x + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{\sinh n\pi x}{\sinh n\pi} \right] \sin n\pi y, \qquad (150)$$

which may be verified easily to satisfy all the boundary conditions. Figure 7.4 shows what the solution looks like.

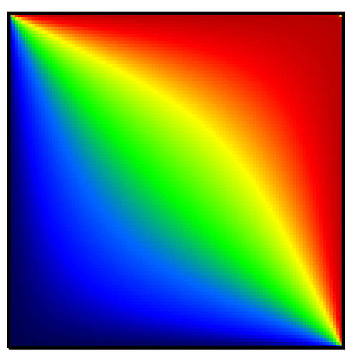


Figure 7.4. Coloured contours for the solution given in Eq. (150) where $\theta = 1$ on $x = 1$ and $y = 1$, while $\theta = 0$ on $x = 0$ and $y = 0$.

8 Solutions of Laplace's equation in polar coordinates.

We shall consider steady-state heat transfer within domains such as circles, sectors of circles (i.e. semicircles), annuli and annular segments. As before, Laplace's equation needs to be solved, but now we need to consider its form in polar coordinates.

Laplace's equation in Cartesian coordinates is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{151}$$

while its polar coordinate counterpart is,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{152}$$

Here, r is the radial coordinate while θ is the angular coordinate in this section of the notes. The big question is: how does one transform Laplace's equation from its Cartesian form into its polar coordinate form? We will now present a derivation but **NOTE** that this derivation is purely for interest's sake and is definitely, very definitely not going to be examined!

The two coordinate systems are related in the following way:

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{153}$$

Now we apply the chain rule but in its two-dimensional form:

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} &&= \cos\theta\,\frac{\partial}{\partial x} + \sin\theta\,\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} &&= -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}.
\end{aligned} \tag{154}$$

These give the partial r and θ derivatives in terms of the x and y derivatives but we need these the other way around. So first we'll rewrite these equations in matrix/vector form:

$$\begin{pmatrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\[2ex] -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} \\[2ex] \dfrac{\partial}{\partial y} \end{pmatrix}. \tag{155}$$

Now we need to solve for the Cartesian derivatives in terms of the polar derivatives, and this will involve multiplying both sides by the inverse of the matrix. We get,

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\[2ex] \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\[2ex] -r\sin\theta & r\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\dfrac{\sin\theta}{r} \\[2ex] \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial \theta} \end{pmatrix}. \tag{156}$$

Finally we are in the position to find expressions for the second Cartesian derivatives. We may write out the first row of Eq. (156) and apply it to itself. This is messy, lengthy and a little intimidating but we get,

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \\
&= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial\theta}\right) \\
&= \boxed{\cos^2\theta\frac{\partial^2 T}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 T}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial T}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 T}{\partial\theta^2}.}
\end{aligned} \tag{157}$$

The corresponding analysis to find the second partial y-derivative is no more difficult but it isn't easier either! We obtain,

$$\frac{\partial^2 T}{\partial y^2} = \boxed{\sin^2\theta\frac{\partial^2 T}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 T}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial T}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial T}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 T}{\partial\theta^2}.} \tag{158}$$

When we add Eqs. (157) and (158) the result is the Laplace operator in polar coordinates:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{becomes} \quad \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} = 0. \tag{159}$$

Now that this has been derived you may forget it, at least for the purposes of the exam at the end of the semester! But while the partial derivatives in the Cartesian version of Laplace's equation have unit coefficients, the polar coordinate version has some r-dependent coefficients. Naturally, this will introduce an extra complexity/novelty but not quite as much as one might fear.

8.1 Typical domains in polar coordinates

In this very short subsection we'll show the types of domain within which we will soon be able to solve Laplace's equation. Essentially they are a circle and aritrary sectors of a circle including the more popular shapes, the semicircle and the quadrant. In addition there are annular versions of all of these, as shown in Fig. 8.1.

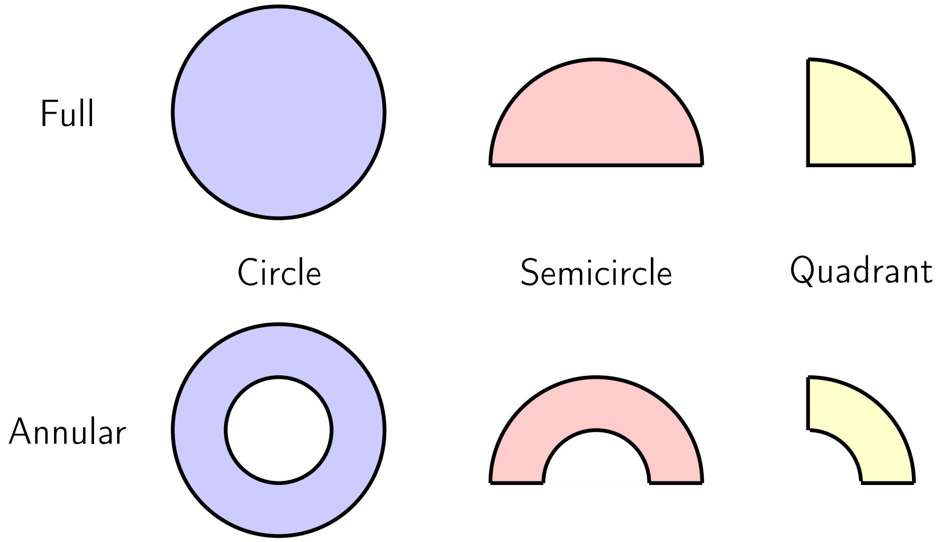


Figure 8.1. Examples of possible domains in polar coordinates.

8.2 Fundamental solutions for heat transfer in a semicircle

We will use the semicircle as an exemplar of the whole separation of variables method as it is adapted to polar coordinate domains, and we will focus on the following system.

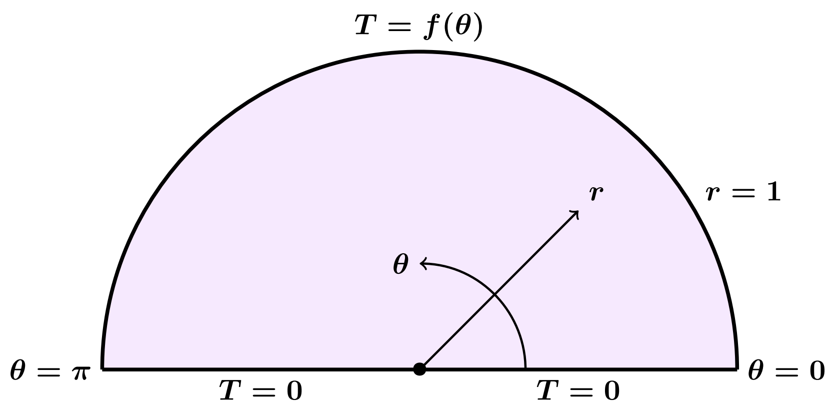


Figure 8.2. Displaying a standard steady 2D heat transfer problem in a semicircle.

We need to notice the following about the above Figure:

- This is a unit semicircle (i.e. the radius is 1);
- The lower surface is held at $T = 0$.
- The curved perimeter at $r = 1$ is held at the given temperature, $T = f(\theta)$.

So how do we even begin to start to solve this? Well, the hint comes from everything we did earlier for the Cartesian problems where we identified a finite range in one coordinate direction into which we could fit sines or cosines in order to satisfy zero boundary condtions. Clearly we cannot do that in the r-direction here because the $r = 1$ boundary condition is that $T = f(\theta)$ is typicaly going to vary with θ and be nonzero. This leaves the θ direction. Now, although the straight boundary looks as though it is just one boundary, it is really two boundaries. One is $\theta = 0$ and the other is $\theta = \pi$, and both of them have $T = 0$ as the boundary condition. Therefore this is direction in which to place our sines.

Ah, but which sines? Well I reckon that $\sin\theta$ works well (i.e. half a sine wave within that range), and so will $\sin 2\theta$, $\sin 3\theta$ and so on. In general we may use $\sin n\theta$. Consequently we shall use the following separation of variables ansatz: let

$$T(r,\theta) = R(r)\sin n\theta \qquad \text{in} \qquad \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{160}$$

Substitution yields,

$$R''\cancel{\sin n\theta} + \frac{1}{r}R'\cancel{\sin n\theta} - \frac{n^2}{r^2}R\cancel{\sin n\theta} = 0, \tag{161}$$

and hence,

$$R'' + \frac{R'}{r} - \frac{n^2}{r^2}R = 0. \tag{162}$$

This equation looks a little better if we multiply by r^2:

$$r^2R'' + rR' - n^2R = 0. \tag{163}$$

At this stage in the analyses when using Cartesian coordinates we obtained linear constant-coefficient ODEs, but this one has r-dependent coefficients. This is an example of what is called a Cauchy-Euler equation and it too has a standard substitution which will yield the solution.

The argument goes as follows, and I am thinking right now of the simultaneous presence of R and rR' in the ODE. Question: what function of r which, when it is differentiated with respect to r and then multiplied afterwards by r, will yield an identically shaped function (although we may ignore constant multiples)? The answer actually turns out to be quite simple: a power of r. Thus a single derivative reduces the power by 1 while the subsequent multiplication by r restores the power to its original value! The upshot is that we may attempt a simple analytical solution of Eq. (163) by substituting,

$$R(r) = Ar^p, \tag{164}$$

where the aim is find suitable values of p. We obtain,

$$Ar^2\Big[p(p-1)r^{p-2}\Big] + Ar\Big[pr^{p-1}\Big] - An^2r^p = 0, \tag{165}$$

which, when tidied up a bit, gives

$$A(p^2 - n^2)r^p = 0. \tag{166}$$

Clearly $A = 0$ is not an option because that yields a zero solution overall which is useless. Likewise the setting of r^p to zero merely tells us that Eq. (166) is satisfied at the origin, and only then if p turns out to be positive! The final option is the Cauchy-Euler equivalent of an Auxiliary (or Indicial or Characteristic) equation, namely,

$$p^2 - n^2 = 0. \tag{167}$$

Therefore we conclude that $p = \pm n$, which provides two potential values for p. We should use both in the first instance and therefore the solution for R is,

$$R = Ar^{-n} + Br^{n}, \tag{168}$$

where A and B are currently arbitrary constants. Given the substitution used in Eq. (160), this means that the fundamental solution for Laplace's equation in polar coordinates is,

$$T(r, \theta) = \Big[Ar^{-n} + Br^{n}\Big] \sin n\theta. \tag{169}$$

Given that Eq. (169) is valid for all positive integer values of n, we may superpose all of them to yield,

$$T(r, \theta) = \sum_{n=1}^{\infty} \Big[A_n r^{-n} + B_n r^{n}\Big] \sin n\theta. \tag{170}$$

Next, we observe that the r^{-n} terms are infinite at the origin. This is unphysical and so they need to be removed by setting $A_n = 0$ for all values of n. This gives,

$$T(r, \theta) = \sum_{n=1}^{\infty} B_n r^{n} \sin n\theta. \tag{171}$$

Then finally we apply the last boundary condition, namely the one at $r = 1$ where $T = f(\theta)$:

$$f(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta, \tag{172}$$

which is a Fourier Sine Series. The Fourier coefficients are given by,

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta. \tag{173}$$

For this semicircular domain that is the end of the analysis where we have an arbitrary temperature profile given on the outer radius. Now we'll consider briefly some specific cases.

8.3 A menagerie of solutions

Example 8.1. Suppose that we choose the temperature at $r = 1$ to be given by $f(\theta) = \theta(\pi - \theta)$. Using the formula for the Fourier coefficients which is given in Eq. (173) we obtain,

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^{\pi} (\pi\theta - \theta^2) \sin n\theta \, d\theta \\ &= \frac{2}{\pi} \Big[\underbrace{\Big(\cancel{\pi\theta - \theta^2}\Big)}_{D_0} \underbrace{\Big(\frac{-\cos n\theta}{n}\Big)}_{I_1} - \underbrace{\Big(\pi - 2\theta\Big)}_{D_1} \underbrace{\Big(\frac{\cancel{-\sin n\theta}}{n^2}\Big)}_{I_2} + \underbrace{\Big(-2\Big)}_{D_2} \underbrace{\Big(\frac{\cos n\theta}{n^3}\Big)}_{I_3} \Big]_0^{\pi} \\ &= \begin{cases} \dfrac{8}{\pi n^3} & \text{for } n \text{ odd,} \\[2ex] 0 & \text{for } n \text{ even.} \end{cases} \end{aligned} \tag{174}$$

Given this expression for B_n the final solution is,

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{\pi n^3} r^n \sin n\theta. \tag{175}$$

As a problem sheet or exam solution, that is as far as we may go, but it is certainly possible to plot the temperature distribution as follows.

Figure 8.3. Depicting the solution given in Eq. (175).

Despite the complexity of the solution process, this final graphic does look a little boring, and it almost looks like a linear temperature profile in the vertical direction. So please indulge my curiosity in this for a moment.

The first two terms of the summation in Eq. (175) are,

$$T = \frac{8}{\pi}\Big(r\sin\theta + \tfrac{1}{27}r^3\sin^3\theta + \cdots\Big) \tag{176}$$

which may be translated back into Cartesians as,

$$T = \frac{8}{\pi}\Big(y + \tfrac{1}{27}(3yx^2 - y^3) + \cdots\Big). \tag{177}$$

The maximum value of the second term is less than 4% of the maximum value of the first term, which means that the second term is generally almost negligible. We see that the first term, when written in Cartesians, is indeed proportional to y, and therefore T varies in a manner which is close to being linear in the vertical direction.

Example 8.2. Find the temperature distribution inside the semicircle when the temperature profile at the outer radius is given by,

$$f(\theta) = \begin{cases} 1 & \text{for } 0 < \theta < \frac{3}{4}\pi, \\ 0 & \text{for } \frac{3}{4}\pi < \theta < \pi \end{cases}. \tag{178}$$

In this case three quarters of the curved boundary is heated, rather than the whole length as we had in Example 8.1.

The solution is again given by Eq. (171) where the Fourier Coefficients, B_n, are given by Eq. (173). Hence,

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} f(\theta)\,\sin n\theta\,d\theta \\
&= \frac{2}{\pi}\left[\int_0^{3\pi/4} 1\,\sin n\theta\,d\theta + \cancel{\int_{3\pi/4}^{\pi} 0\,\sin n\theta\,d\theta}\right] \\
&= \frac{2}{\pi}\left[-\frac{\cos n\theta}{n}\right]_0^{3\pi/4} \\
&= \frac{2}{\pi}\left[\frac{1-\cos\frac{3n\pi}{4}}{n}\right],
\end{aligned} \tag{179}$$

and therefore the final solution is,

$$T = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-\cos\frac{3n\pi}{4}}{n}\right] r^n \sin n\theta. \tag{180}$$

This temperature distribution looks like:

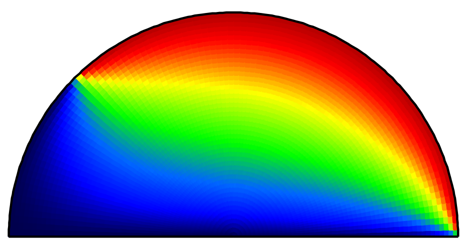


Figure 8.4. Depicting the solution given in Eq. (180).

Figure 8.4 shows the extent to which the cold region extends into the semicircle when only three quarters of its outer radius is heated.

Example 8.3. A case when only the middle $1/5^{\text{th}}$ of the curved boundary is heated.

The outer temperature profile is,

$$f(\theta) = \begin{cases} 0 & \text{for } 0 < \theta < \frac{2}{5}\pi, \\ 1 & \text{for } \frac{2}{5}\pi < \theta < \frac{3}{5}\pi \\ 0 & \text{for } \frac{3}{5}\pi < \theta < \pi \end{cases}, \tag{181}$$

and the Fourier Coefficients may be obtained in exactly the same way as for the previous example. Omitting the details of the integration, we obtain the solution,

$$T = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos\frac{2n\pi}{5}-\cos\frac{3n\pi}{5}}{n}\right] r^n \sin n\theta, \tag{182}$$

and this temperature field looks like the following.

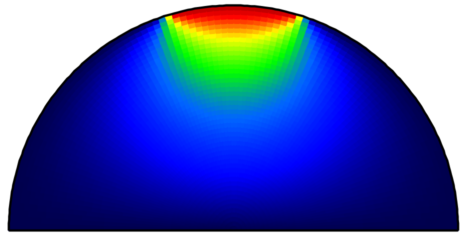


Figure 8.5. Depicting the solution given in Eq. (180).

Thus the effect of having only a short heating length is that that heat does not penetrate very at all into the semicircle.

Example 8.4. Referring back to Eq. (170), reproduced here for convenience,

$$T(r,\theta) = \sum_{n=1}^{\infty} \Big[A_n r^{-n} + B_n r^n \Big] \sin n\theta, \tag{183}$$

the very next step in the analysis there was the removal of the A_n terms in order to avoid infinities at the origin. But if we were to replace the r^n terms in Eq. (182) by the corresponding r^{-n} terms to get,

$$T = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{\cos\frac{2n\pi}{5} - \cos\frac{3n\pi}{5}}{n} \right] r^{-n} \sin n\theta, \tag{184}$$

then what meaning does this have?

We note:

- The first observation is that it too satisfies Laplace's equation. The reason is that both the $r^n \sin n\theta$ term in Eq. (182) and the $r^{-n} \sin n\theta$ term here satisfy Laplace's equation.
- The second observation is that $T = 0$ when $\theta = 0$ and when $\theta = \pi$. This is due to the continuing presence of the $\sin n\theta$ term.
- The substitution of $r = 1$ into Eqs. (182) and (184) yields identical expressions, namely the Fourier Sine series of $f(\theta)$ in (181). So the boundary conditions are identical for the two cases.
- Although Eq. (184) is infinite at the origin, it is nevertheless finite as $r \to \infty$.

So the facts that Eq. (184) (i) satisfies the same boundary conditions as Example 8.3 does; (ii) satisfies Laplace's equation and (iii) is finite as $r \to \infty$, lead us to the conclusion that this solution represents the temperature field which is outside of the semicircle, as shown in the Figure 8.6 below.

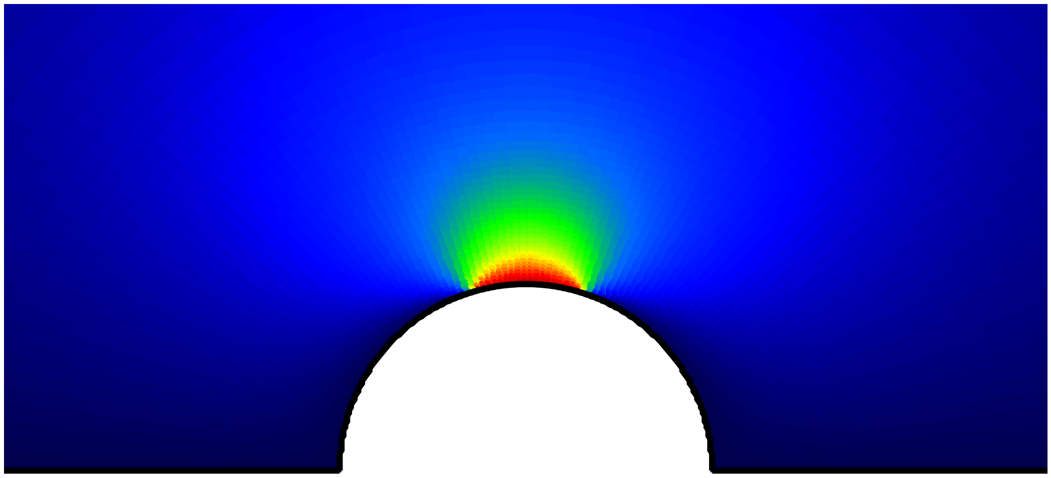


Figure 8.6. Depicting the solution given in Eq. (184).

Even in this external configuration (external to the semicircle, that is) the heat still doesn't penetrate very far away from the semicircle. In fact the penetration distance is approximately the same as the width of the hot spot.

The main conclusion from this example is that this technique may also be used to solve certain external heat transfer problems.

Example 8.5. We will maintain the semicircular theme for a little longer but now the $\theta = \pi$ boundary is insulated.

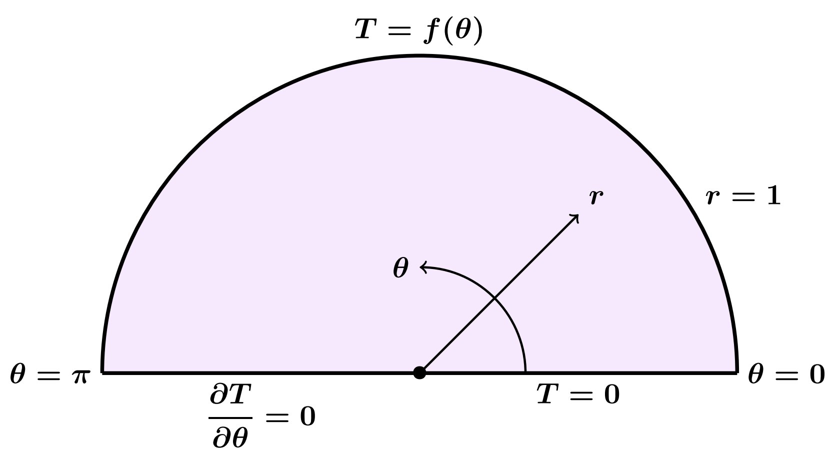


Figure 8.7. Displaying the present semicircular domain with an insulated boundary at $\theta = \pi$.

The change in this boundary condition from all of the previous examples means that we need to rework the fundamental solutions part of the analysis. The new boundary condition at $\theta = \pi$ now suggests that we should use $\sin \frac{1}{2} n\theta$ with odd values of n as the angular components of the solution. A quick sketch in the range $0 \leq \theta \leq \pi$ will enough to show that the sines are all equal to zero when $\theta = 0$ and that their derivatives are zero at $\theta = \pi$ when n is odd. Thus we are going to be obtain a solution using a Quarter-range Fourier Series.

Beginning again with the steady 2D Laplace's equation in polar coordinates,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0, \tag{185}$$

we will let

$$T(r,\theta) = R(r)\sin\tfrac{1}{2}n\theta, \tag{186}$$

and hence,

$$R''\cancel{\sin\tfrac{1}{2}n\theta} + \frac{1}{r}R'\cancel{\sin\tfrac{1}{2}n\theta} - \frac{n^2}{4r^2}R\cancel{\sin n\theta} = 0. \tag{187}$$

Therefore R satisfies,

$$R'' + \frac{R'}{r} - \frac{n^2}{4r^2}R = 0 \qquad \text{or} \qquad r^2R'' + rR' - \tfrac{1}{4}n^2R = 0. \tag{188}$$

The substitution of $R = r^p$ yields,

$$(p^2 - \tfrac{1}{4}n^2)r^p = 0, \tag{189}$$

and so $p = \pm\frac{1}{2}n$. This leads to

$$R = Ar^{-n/2} + Br^{n/2}, \tag{190}$$

where A and B are arbitrary constants. Reconstruction of T followed by superposition yields,

$$T(r,\theta) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[A_n r^{-n/2} + B_n r^{n/2}\right]\sin\tfrac{1}{2}n\theta. \tag{191}$$

Given that we are concerned with the interior of the semicircle we may set $A_n = 0$ for all values of n, and therefore,

$$T(r,\theta) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B_n r^{n/2}\,\sin\tfrac{1}{2}n\theta. \tag{192}$$

Then finally we apply the last boundary condition, namely the one at $r = 1$ where $T = f(\theta)$:

$$f(\theta) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B_n \sin\tfrac{1}{2}n\theta, \tag{193}$$

which is the Quarter-range Fourier Series that we expected. The Fourier coefficients are given by,

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\,\sin\tfrac{1}{2}n\theta\,d\theta. \tag{194}$$

This completes the separation of variables analysis for this Example and all that remains is to choose a boundary temperature profile, find the Fourier coefficients and write out the solution.

For this example we will set $f(\theta) = 1$, a uniform boundary temperature. The Fourier coefficients are,

$$B_n = \frac{2}{\pi}\int_0^{\pi} 1\,\sin\tfrac{1}{2}n\theta\,d\theta = \frac{4}{n\pi} \quad (n \text{ odd}). \tag{195}$$

Hence the internal temperature distribution is,

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} r^{n/2} \sin \tfrac{1}{2} n\theta, \tag{196}$$

noting again that quarter-range series are confined to odd values of n. The temperature distribution is shown in Fig. 8.8.

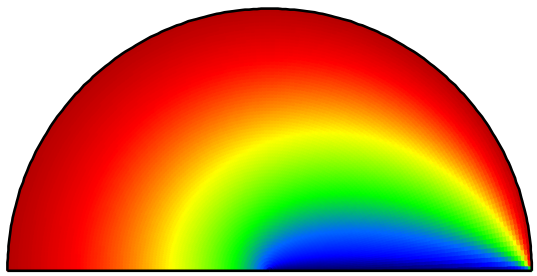


Figure 8.8. Depicting the solution given in Eq. (196).

For the sake of comparison, the equivalent (Fourier Sine Series) solution for the case when the whole of the straight boundary is maintained at $T = 0$ is,

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} r^{n} \sin n\theta, \tag{197}$$

which is surprisingly similar to Eq. (196), a Quarter-range Fourier Series, and the temperature distribution has the form,

Example 8.6. Other sectors of a circle.

We shall now extend the above work on semicircles to other sectors of a circle:

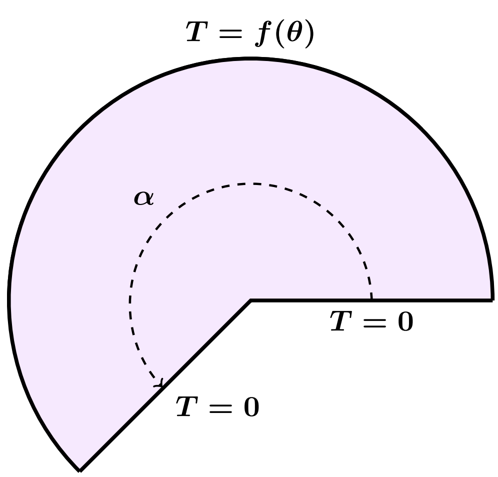


Figure 8.9. Depicting a sector which subtends an angle of α.

When $T = 0$ is imposed on both the $\theta = 0$ and $\theta = \alpha$ boundaries, then we will need to use sines of the form, $\sin(n\pi\theta/\alpha)$ in the separation of variables ansatz. One may derive this in two different ways, and both involve beginning with the $n = 1$ basic half sine wave.

First we note that θ/α varies between 0 and 1 while θ varies between 0 and α. Therefore $\pi\theta/\alpha$ varies between 0 and π, and this covers half a sine wave, as desired. So we should use $n\pi\theta/\alpha$ as the argument to the sine.

The second route is to start with $\sin c\theta$, where c is to be found. Now we insist that $c\theta$ must vary between 0 and π while θ varies between 0 and α. Therefore $c\alpha = \pi$ and so $c = \pi/\alpha$. Therefore one half sine wave is represented by $\sin \pi\theta/\alpha$, and n half sine waves are then represented by $\sin n\pi\theta/\alpha$.

For the general angle, α, we let

$$T(r,\theta) = R(r)\sin\left(\frac{n\pi\theta}{\alpha}\right), \tag{198}$$

and hence,

$$R''\cancel{\sin\left(\frac{n\pi\theta}{\alpha}\right)} + \frac{1}{r}R'\cancel{\sin\left(\frac{n\pi\theta}{\alpha}\right)} - \frac{n^2}{4r^2}R\cancel{\sin\left(\frac{n\pi\theta}{\alpha}\right)} = 0. \tag{199}$$

Therefore R satisfies,

$$R'' + \frac{R'}{r} - \frac{n^2\pi^2}{\alpha^2 r^2}R = 0 \qquad \text{or} \qquad r^2R'' + rR' - \frac{n^2\pi^2}{\alpha^2}R = 0. \tag{200}$$

The substitution of $R = r^p$ yields,

$$\left(p^2 - \frac{n^2\pi^2}{\alpha^2}\right)r^p = 0, \tag{201}$$

and so $p = \pm n\pi/\alpha$. This leads to

$$R = Ar^{-n\pi/\alpha} + Br^{n\pi/\alpha}, \tag{202}$$

where A and B are arbitrary constants. Reconstruction of T followed by superposition yields,

$$T(r,\theta) = \sum_{n=1}^{\infty}\Big[A_n r^{-n\pi/\alpha} + B_n r^{n\pi/\alpha}\Big]\sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{203}$$

Given that we shall be solving within the domain shown in Fig. 8.9, we have to set $A_n = 0$ in order not to have infinite solutions at the origin. This yields,

$$T(r,\theta) = \sum_{n=1}^{\infty} B_n r^{n\pi/\alpha}\sin\left(\frac{n\pi\theta}{\alpha}\right). \tag{204}$$

Finally, we need to apply the boundary condition that $T = f(\theta)$ at $r = 1$; hence

$$T(r,\theta) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi\theta}{\alpha}\right), \tag{205}$$

where the Fourier Sine Series coefficients are given by,

$$B_n = \frac{2}{\alpha} \int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta. \tag{206}$$

Clearly this formula reduces to the one given in Eq. (173) when the sector becomes a semicircle again when $\alpha = \pi$.

We will complete this example by considering a quadrant where the curved boundary at $r = 1$ is held at the uniform temperature, $T = f(\theta) = 1$.

The solution for a quadrant is given by Eqs. (205) and (206) with $\alpha = \frac{1}{2}\pi$ and $f(\theta) = 1$. The value of B_n is given by,

$$B_n = \frac{4}{\pi} \int_0^{\pi/2} 1 \sin 2n\theta \, d\theta = \frac{4}{\pi} \left[\frac{-\cos 2n\theta}{2n}\right]_0^{\pi/2} = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \tag{207}$$

Hence T is given by

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} r^{2n} \sin 2n\theta, \tag{208}$$

and the temperature field is represented in the following Figure.

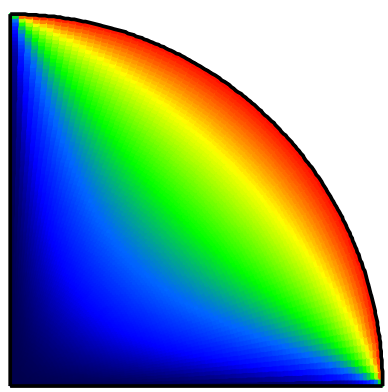


Figure 8.10. Depiction the solution given in Eq. (208).

It is clear that the heat gained from the curved surface doesn't penetrate very far into the quadrant. All the heat which is transferred into the quadrant via the relatively short hot surface then exits from the quadrant via the relatively long cold surface. Thus the temperature gradient at the curved boundary will be much larger than that on the straight boundaries, and this manifests itself as a thermal boundary layer near to $r = 1$.

Example 8.7. Steady conduction in a sector which subtends the angle $\frac{5}{4}\pi$.

Following exactly the same procedure as on the previous page, but with $\alpha = \frac{5}{4}\pi$, the Fourier Sine Series coefficient is,

$$B_n = \frac{2}{(\frac{5}{4})\pi} \int_0^{5\pi/4} 1 \sin \tfrac{4}{5} n\theta \, d\theta = \frac{8}{5\pi} \left[\frac{-\cos \frac{4}{5} n\theta}{\frac{4}{5} n} \right]_0^{4\pi/5} = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \tag{209}$$

Hence T is given by

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} r^{4n/5} \sin \tfrac{4}{5} n\theta, \tag{210}$$

and the temperature profile is shown below.

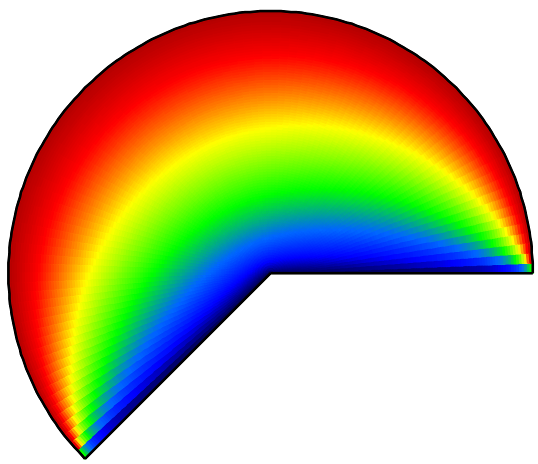


Figure 8.11. Depiction of the solution given in Eq. (210).

Now that the heated surface is much longer than in Figure 8.10 and, indeed, is longer than the cold surface a greater proportion of the sector may be regarded as being hot.

Example 8.8. This case represents the most extreme version of a sector by occupying the range $0 \leq \theta \leq 2\pi$. But this isn't a full circle but rather a full circle with a cold fin at $\theta = 0$ and $\theta = 2\pi$. Quite how this configuration could be set up is beyond me, but at least we can calculate what the temperature would be if we could! Therefore we let $\alpha = 2\pi$ in the earlier general analysis. The final solution may be found easily to be,

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} r^{n/2} \sin \tfrac{1}{2} n\theta \tag{211}$$

and the temperature field is:

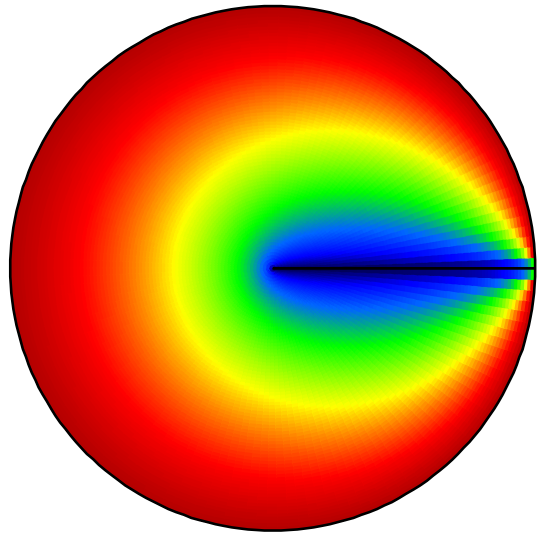
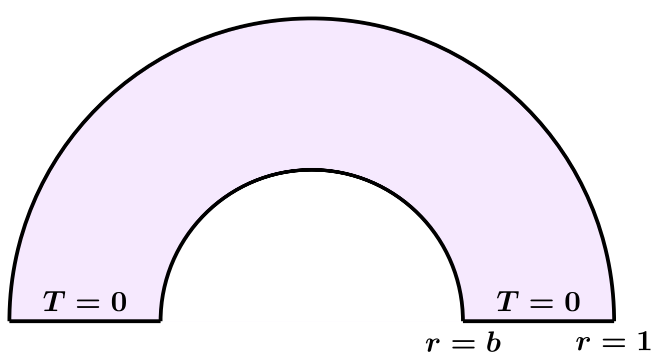


Figure 8.12. Depiction of the solution given in Eq. (211).

This may look somewhat familiar. Indeed Eq. (211) is precisely the same as Eq. (196), which was a Quarter-range Fourier Series in a semicircle. The reason that the formulae are identical is that the present boundary temperature, $f(\theta) = 1$, is symmetrical about $\theta = \pi$ and this means that the solution itself has the same symmetry. Given that this even symmetry is equivalent to having $\partial\theta/\partial x = 0$, and that that is the boundary condition which was imposed in Example 8.5, it is to be expected that the solutions are identical.

Note: If one can detect in advance that a solution should be symmetric, then this means that short-cuts may be made for the corresponding numberical solutions. So for example, if I were to have given Example 8.8 as a numerical problem to solve, and it were noticed that the solution should have even symmetry about $\theta = \pi$, then it would be advantageous to solve Example 8.5 instead. Just to say that, technically, the $\partial T/\partial\theta = 0$ condition at $\theta = \pi$ is an insulating condition in Example 8.5 whereas it would used as a symmetry condition if we used Example 8.5 to emulate Example 8.8. The advantage of all of this is that the number of grid points is halved, while the number of iterations may well be fewer than half, and this leads a compuation time of less than 25% of the full problem.

Example 8.9. Going annular! In this final example (which will comprise two very slightly different versions) we will consider a semicircular annulus.



We'll consider a semicircular annulus which occupies the region, $b \le r \le 1$ and $0 \le \theta \le \pi$, we shall treat this particular cavity as an exemplar. Back in Eq. (170) we derived the following expression for the fundamental solution for a semicircular domain where $T = 0$ on both $\theta = 0$ and $\theta = \pi$:

$$T(r, \theta) = \sum_{n=1}^{\infty} \Big[A_n r^{-n} + B_n r^n \Big] \sin n\theta. \tag{212}$$

This solution also applies here if the straight boundaries of the annulus correspond to $T = 0$.

In this example the origin is no longer part of the system and this means that we cannot set $A_n = 0$ to avoid the solution becoming infinite at the origin. Similarly, the domain is finite and therefore we have no reason to set $B_n = 0$, which would avoid the solution becoming infinite as $r \to \infty$. Se we need to keep both of these coefficients. We have seen this before in section 7 when we solved Laplace's equation in square domains.

In the first instance We will adopt the following boundary conditions on $r = b < 1$ and $r = 1$.

$$T = f(\theta) \text{ on } r = b \quad \text{and} \quad T = 0 \text{ on } r = 1. \tag{213}$$

For simplicity with a hint of generality we shall assume that the Fourier Sine Series of $f(\theta)$ is

$$f(\theta) = \sum_{n=1}^{\infty} C_n \sin n\theta. \tag{214}$$

Now we apply the boundary condition at $r = 1$. This yields,

$$0 = \sum_{n=1}^{\infty} \Big[A_n + B_n \Big] \sin n\theta \quad \Longrightarrow \quad A_n + B_n = 0. \tag{215}$$

On applying the boundary conditions at $r = b$ we obtain,

$$\sum_{n=1}^{\infty} C_n \sin n\theta = \sum_{n=1}^{\infty} \Big[A_n b^{-n} + B_n b^n \Big] \sin n\theta \quad \Longrightarrow \quad A_n b^{-n} + B_n b^n = C_n. \tag{216}$$

We end up with a pair of simultaneous equations for A_n and B_n. This gives,

$$B_n = \frac{C_n}{b^n - b^{-n}} = -A_n, \tag{217}$$

and therefore the final solution is,

$$T = \sum_{n=1}^{\infty} C_n \left[\frac{r^n - r^{-n}}{b^n - b^{-n}} \right] \sin n\theta. \tag{218}$$

Despite its complexity, it is quite straightforward to check that this solution satisfies the boundary conditions. When $r = 1$ the numerator of the quotient is zero. When $r = b$ the term in the square brackets is precisely 1 and we reproduce Eq. (214), which is the Fourier Sine Series of the boundary temperature profile.

This solution may be illustrated by choosing $b = 1/2$ and with $f(\theta) = 1$. Equation (218) becomes,

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{(b/r)^n - (r/b)^n}{b^n - (1/b)^n} \right] \sin n\theta, \tag{219}$$

and the following Figure shows the temperature distribution.

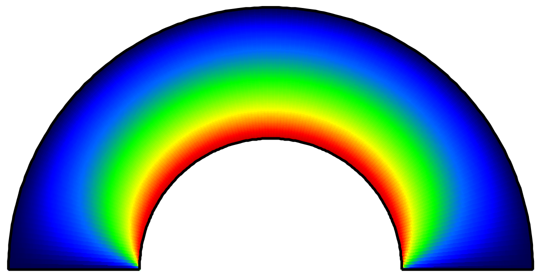


Figure 8.13. Depiction of the solution given in Eq. (219) for inner heating with $b = \frac{1}{2}$.

We may also consider what happens if it is the outer boundary which is heated. Omitting the detailed analysis we obtain,

$$T = \sum_{n=1}^{\infty} C_n \left[\frac{(b/r)^n - (r/b)^n}{b^n - (1/b)^n} \right] \sin n\theta, \tag{220}$$

for the general case, and

$$T = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \left[\frac{(b/r)^n - (r/b)^n}{b^n - (1/b)^n} \right] \sin n\theta, \tag{221}$$

when $T = 1$ on $r = 1$.

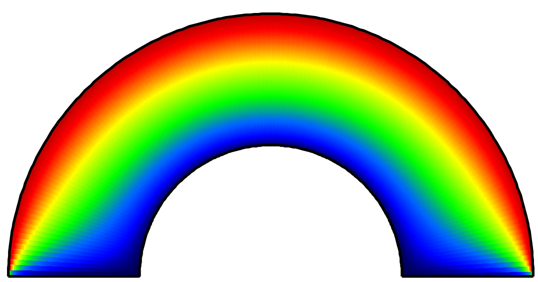


Figure 8.14. Depiction of the solution given in Eq. (220) for outer heating with $b = \frac{1}{2}$.

Finally, if it were the case that the inner curved boundary has the temperature profile, $T = f(\theta)$ and the outer curved surface were held at $T = g(\theta)$ where

$$f(\theta) = \sum_{n=1}^{\infty} F_n \sin n\theta \qquad \text{and} \qquad g(\theta) = \sum_{n=1}^{\infty} G_n \sin n\theta, \tag{222}$$

then we may add suitably modified versions of Eqs. (218) and (220) to yield

$$T = \sum_{n=1}^{\infty} F_n \left[\frac{r^n - r^{-n}}{b^n - b^{-n}} \right] \sin n\theta + \sum_{n=1}^{\infty} G_n \left[\frac{(b/r)^n - (r/b)^n}{b^n - (1/b)^n} \right] \sin n\theta. \tag{223}$$

8.4 A brief summary of solutions of Laplace's equations in polar coordinates.

Generally we substitute $\boxed{T = R(r)\sin n\gamma\theta}$ where γ depends on the BCs and the subtended angle. This is for either a Fourier Sine Series or a Quarter-range Series. The equivalent cosines would need to be used for insulating boundary conditions.

We always obtain $\boxed{r^2R + rR' - n^2\gamma^2 R = 0}$ for which $\boxed{R = Ar^{n\gamma} + Br^{-n\gamma}}$ is the solution.

Hence:
$$\boxed{T = \sum_{n=1}^{\infty}\Big[A_n r^{-n\gamma} + B_n r^{n\gamma}\Big]\sin n\gamma\theta.}$$

(i) Internal domains: $\boxed{A_n = 0.}$ (ii) External domains: $\boxed{B_n = 0.}$ (iii) Annuli: $\boxed{\text{neither are zero.}}$

BCs are then imposed which give (i) B_n, (ii) A_n and (iii) simultaneous equations for A_n and B_n, respectively.

Finally, circular domains require full Fourier Series, and we get

$$\boxed{T = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\left[A_n\cos n\theta + B_n\sin n\theta\right]} \qquad \text{inside a circle}$$

and

$$\boxed{T = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}\left[A_n\cos n\theta + B_n\sin n\theta\right]} \qquad \text{outside a circle.}$$

9 The standard textbook method of Separation of Variables

Note: The analysis contained in this section is not required for `ME20021`, but is included for information only because it is this analysis which is always presented in textbooks. In my view, and particularly for the PDEs that we are solving, it is unwieldy and overly long. However, it is worth having a quick glance through this so that you're not surprised when reading a textbook!

So I will illustrate the standard textbook method by repeating the analysis of §2.1, the solution of Fourier's equation,

$$\frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}, \tag{224}$$

in the domain, $0 \le x \le 1$, and subject to $\theta = 0$ on both $x = 0$ and $x = 1$.

First we **assume** a separation-of-variables solution, $\theta(x,t) = X(x)T(t)$, where neither X nor T are known. This is substituted into Eq.(224) to get,

$$XT' = \alpha X''T. \tag{225}$$

Note that I have used primes to denote ordinary derivatives with respect to the appropriate variable, so X' means dX/dx and T' means dT/dt. If we now divide both sides of Eq. (225) by XT we obtain,

$$\frac{T'}{T} = \alpha\frac{X''}{X}. \tag{226}$$

It turns out to be a little more convenient to divide both sides by α as well:

$$\frac{T'}{\alpha T} = \frac{X''}{X}. \tag{227}$$

This innocent-looking equation has important ramifications. The left hand side is a function of t only, and we would normally expect that any change in the value of t would alter the value of the left hand side. However, the right hand side is a function of x only, and therefore it must be unaffected by changes in the value of t. This means that the left hand side cannot vary as t varies. A similar argument means that changes in x will not affect the value of either side of the equation. Therefore both sides must be equal to a constant, which we will take to be $\mathcal{K}$; this is called the **separation constant**. Equation (227) becomes,

$$\frac{T'}{\alpha T} = \frac{X''}{X} = \mathcal{K}, \tag{228}$$

and therefore we must have both

$$X'' - \mathcal{K}X = 0 \qquad \text{and} \qquad T' - \mathcal{K}T = 0. \tag{229}$$

Further progress now requires us to consider the physics and/or the mathematics of the problem in hand in order to choose not only the sign of $\mathcal{K}$ but also suitable values of it. There are three cases to consider and, in the time-honoured tradition of academic exegesis, we will consider the correct one last!

Case 1. Let us take $\mathcal{K} = p^2 > 0$. Then the solutions to Eqs. (229) are

$$X = Ae^{px} + Be^{-px} \qquad \text{and} \qquad T = Ce^{\alpha p^2 t}, \tag{230}$$

where A, B and C are arbitrary constants.

Now we shall apply the boundary conditions. If $\theta = 0$ at $x = 0$ and $x = 1$, then $X = 0$ at these points too. Therefore the setting of $x = 0$ into Eq. (230) gives

$$0 = A + B, \qquad \Rightarrow \qquad B = -A, \tag{231}$$

while the setting of $x = 1$ gives

$$0 = Ae^{p} + Be^{-p} \qquad \Rightarrow \qquad B = -Ae^{2p}. \tag{232}$$

The elimination of B between Eqs. (231) and (232) shows that $B = 0$ and hence that $A = 0$. In turn this implies that $X = 0$ and therefore $\theta = 0$. While this solution is undoubtedly correct in the sense that the equations and boundary conditions which have been applied so far are satisfied, it is not particularly useful. However, we could have circumvented this mathematical analysis by noting that we must set $C = 0$ in Eq. (230) because we cannot have exponentially growing solutions in time in such a heat transfer problem. This physical observation means that $T = 0$ and hence that $\theta = 0$ once more.

Case 2. Let us now take $\mathcal{K} = 0$. Equations (229) reduce to

$$X'' = 0 \qquad \text{and} \qquad T' = 0. \tag{233}$$

The solutions are that

$$X = Ax + B \qquad \text{and} \qquad T = C. \tag{234}$$

Although $C =$ constant is physically reasonable, the application of the boundary conditions again yields $A = B = 0$, implying that $\theta = 0$. Thus this choice of the separation constant is no good either.

Case 3. Finally, let us take $\mathcal{K} = -p^2$. Equations (229) become,

$$X'' + p^2 X = 0 \qquad \text{and} \qquad T' + \alpha p^2 T = 0. \tag{235}$$

The solutions are that

$$X = A \cos px + B \sin px \qquad \text{and} \qquad T = Ce^{-\alpha p^2 t}, \tag{236}$$

and from this we get

$$\theta(x,t) = (A \cos px + B \sin px)Ce^{-\alpha p^2 t}. \tag{237}$$

This equation looks as though it has three arbitrary constants, A, B and C, but in fact it has only two: AC and BC, and therefore we may rewrite Eq. (237) as

$$\theta(x,t) = (A \cos px + B \sin px)e^{-\alpha p^2 t}. \tag{238}$$

At $x = 0$ we have $\theta = 0$ and substitution of this into (238) gives $A = 0$. Our solution is now,

$$\theta(x,t) = B \sin px \; e^{-\alpha p^2 t}. \tag{239}$$

At $x = 1$ we also have $\theta = 0$ and Eq. (239) yields

$$0 = B \sin p \; e^{-\alpha p^2 t}. \tag{240}$$

Clearly we must not take $B = 0$ because this again leaves us with $\theta = 0$. Therefore we must take

$$\sin p = 0, \tag{241}$$

from which we deduce that p must be a multiple of π. So we shall take

$$p = n\pi \quad \text{for any positive integer value of } n. \tag{242}$$

Using this information in Eq. (239) means that our solution is

$$\theta(x,t) = B \sin n\pi x \; e^{-\alpha n^2 \pi^2 t}, \tag{243}$$

for any integer value of n. The good news is that we have just arrived at Eq. (12) and therefore all that remains is to superpose all of the possible solutions; we get

$$\theta(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x \; e^{-\alpha n^2\pi^2 t}, \tag{244}$$

and to apply whatever the initial condition is at $t = 0$.

To conclude, this long-winded way of applying separation of variables had taken more than two pages of dense argument, whereas the approach which has been taken in earlier sections used only a one or two line ansatz where the appropriate sine or cosine was chosen to satisfy the boundary conditions. To be fair, there are other PDEs for which the long-winded approach turns out to be necessary because it isn't obvious what specific function needs to be used in a separation-of-variables ansatz. Fortunately we have no need of this in `ME20021`.

Fourier Transforms for solving Partial Differential Equations

10 Introduction

10.1 From Fourier series to Fourier transforms

We have already seen how Fourier Series may be used to solve PDEs. Crudely speaking, the way we did it was to use a suitable selection of solutions of the form,

$$\theta = B \sin kx \, e^{-ky} \tag{245}$$

(which is an appropriate one for solving Laplace's equation) for suitable values of k, adding them all together (superposing), and finally finding the values of the arbitrary constants using Fourier Series methods. Therefore a Laplace's equation problem in the following domain,

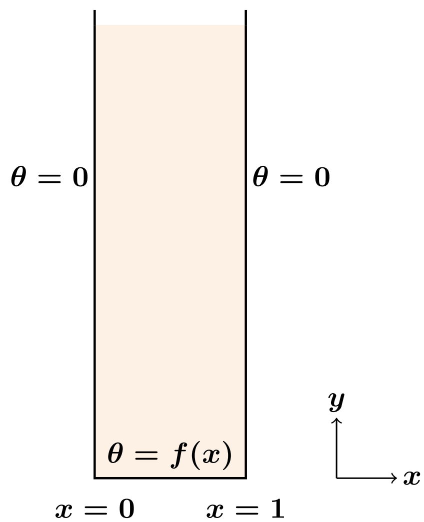


Figure 10.1. Fourier series may be used to solve Laplace's equation in domains which are of finite width in one direction.

has a solution of the form,

$$\theta = \sum_{n=1}^{\infty} B_n \sin n\pi x \, e^{-n\pi y}. \tag{246}$$

The domain shown above in Figure 10.1 is finite in the x-direction, and therefore Fourier Series may be used in that direction. However, when the domain is infinitely wide, i.e. $-\infty < x < \infty$, then this is the type of problem for which Fourier Transforms need to be used; see Fig. 2 on the next page. Instead of modifying Eq. (245) using summations over a set of discrete frequencies to obtain Eq. (246), Eq. (245) may be modified by integrating over a continuous set of frequencies, k:

$$\theta = \int_0^{\infty} F(k) \, \sin ky \, e^{-ky} \, dy. \tag{247}$$

We may also consider $F(k)$ in (247) to be a continuous function of wavenumber and therefore to be the analogue counterpart to the digitial B_n in (246).

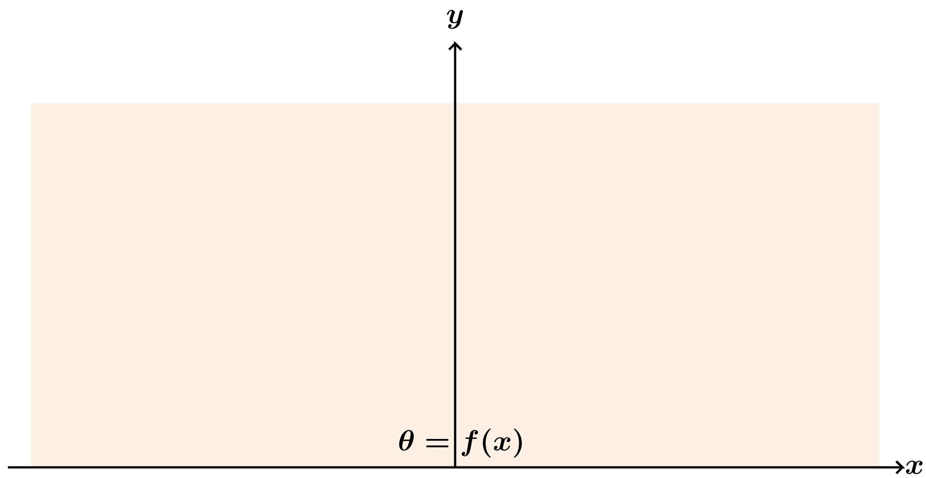


Figure 10.2. Fourier Transforms may be used to solve Laplace's equation in domains which are of infinite width in one direction.

As we will see later, the formula given in Eq. (247) forms one special imstance and a more general formula is the following,

$$\theta = \frac{2}{\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}e^{-|\omega|y}\,d\omega, \tag{248}$$

where ω is a more-frequently used notation for the wavenumber.

To summarise the distinction verbally, a Fourier Series is a sum over a set of discrete frequencies whil a Fourier Transform is an integral over continuous set of frequencies.

10.2 Definition of the Fourier Transform and its inverse.

The so-called Fourier Transform pair is defined this way,

$$\mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx = F(\omega), \tag{249}$$

$$\mathcal{F}^{-1}\left[F(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}\,d\omega = f(x). \tag{250}$$

To be safe, I confirm that if one wishes to speak about $\mathcal{F}\left[f(x)\right]$ then the correct way is to say

"Fourier Transform of $f(x)$...."

and we may also say that,

"$\mathcal{F}[\ \ ]$ is the Fourier Transform operator"

in the same as we may say that d/dx is a differential operator.

In Eq. (249) note that the presence of the complex exponential in the integral means that the Fourier Transform of a real $f(x)$ will usually yield a complex-valued $F(\omega)$.

Note also that ω is a spatial frequency, and therefore the Fourier Transform of a function gives us its frequency content. I will illustrate this in much more detail later.

Finally, note that the formula for the inverse Fourier Transform bears a lot of similarity to the definition of the Fourier Transform itself. The two differences are (i) the sign of the exponent in the complex exponential, and (ii) the different constants multiplying the integrals, although some textbooks use $1/\sqrt{2\pi}$ for both integrals, something I don't like because then $\sqrt{\pi}$ appears everywhere like mud and road dirt on a bicycle.

The formulae for the Fourier Transform pair will be quoted on the exam paper.

10.3 Comparison with the Laplace Transform

Last year (`ME10305` Mathematics 2)I made the very strange statement that the Laplace Transform variable, s, could be interpreted as an imaginary frequency. In practical terms this makes no sense! But when we look at and compare the two formulae:

$$\text{Laplace Transform:} \quad \mathcal{L}\left[f(x)\right] = \int_0^\infty f(x)e^{-sx}\,dx$$

$$\text{Fourier Transform:} \quad \mathcal{F}\left[f(x)\right] = \int_{-\infty}^\infty f(x)e^{-j\omega x}\,dx,$$

we see that s plays the same sort of role in Laplace transforms as does $j\omega$ in Fourier Transforms. Given that one always thinks of s as being real, then in a sense this is equivalent to ω being purely imaginary.

Having said all of that, you may now forget it!

10.4 Existence of the Fourier Transform

The main difference between the Laplace Transform and the Fourier Transform is that the range of functions for which the Fourier Transform integral gives us a finite value, i.e. a well-defined function, is smaller than for a Laplace Transform. We can find $\mathcal{L}\left[x\right]$ but not $\mathcal{F}\left[x\right]$. The former is $1/s^2$ (assuming that $s > 0$), but the integral for the latter does not converge. This is all due to the fact that e^{-sx} decays exponentially as x becomes large, while the real and imaginary parts of $e^{-j\omega x}$ oscillate between $+1$ and -1. Therefore there is a set of rules to determine in advance whether or not the Fourier Transform integral converges to something sensible. Here they are....

1. **$f(x) \to 0$ as $x \to \pm\infty$**

2. **$f(x)$ must be finite everywhere.**

Note the strange wording: these are **sufficient** conditions. This means that a Fourier Transform is guaranteed to exist if both conditions are satisfied, although that doesn't guarantee that the integrals may be obtained analytically. However, there remain some exotic functions which violate one or other of these conditions but yet still have a Fourier Transform.

The following Table gives some example functions.

$f(x)$	✓/✗	$F(\omega)$	Comment	
e^{-x^2}	✓	$\sqrt{\pi}e^{-\omega^2/4}$		
$e^{-\lvert x\rvert}$	✓	$2/(1+\omega^2)$		
$1/(1+x^2)$	✓	$\pi e^{-\lvert\omega\rvert}$		
unit pulse	✓			
e^{-x}	✗		violates (i)	$\longrightarrow \infty$ as $x \longrightarrow -\infty$
$1/x^2$	✗		violates (ii)	infinite at $x = 0$
1	✓	$2\pi\delta(\omega)$	violates (i)	
$\cos ax$	✓	$\pi[\delta(\omega-a)+\delta(\omega+a)]$	violates (i)	
$\lvert x\rvert^{-1/2}$	✓	$\sqrt{2\pi}\lvert\omega\rvert^{-1/2}$	violates (ii)	
$\cos(x^2)$	✓	$\sqrt{\pi}\cos(\frac{1}{4}\omega^2 - \frac{1}{4}\pi)$	violates (i)	

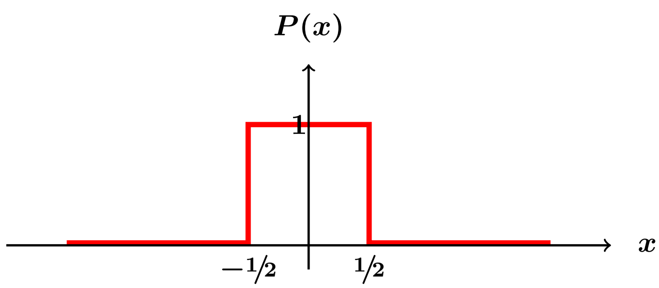
The first set of functions satisfy both conditions and therefore have Fourier Transforms. The transform of the unit pulse is not given because it depends on its duration and where it is centred. The second set, a pair, violates one condition each and don't have transforms. The third set is in that fuzzy region between the first two where a sufficient condition is violated but there is, nevertheless, a transform. It is interesting to note that, if those transforms were converted back to functions of x simply by replacing ω with x, then they too will violate one of the conditions.

In many ways this subsection was really just for information only.

11 Examples of Fourier Transforms

11.1 Example 1.

We shall find the Fourier Transform of the unit pulse of duration 1 which is centred at $x = 0$.



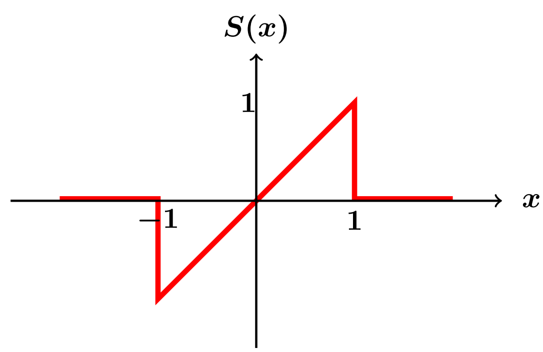
In the following derivation of the Fourier Transform of this unit pulse, use is made of the fact that it is an even function, and that integrals of odd functions (over symmetric intervals) are zero.

$$
\begin{aligned}
\mathcal{F}\left[P(x)\right] &= \int_{-\infty}^{\infty} P(x)e^{-j\omega x}\,dx && \text{by definition}\\
&= \int_{-\infty}^{-1/2} 0\,e^{-j\omega x}\,dx + \int_{-1/2}^{1/2} 1\,e^{-j\omega x}\,dx + \int_{1/2}^{\infty} 0\,e^{-j\omega x}\,dx && \text{splitting into three regions}\\
&= \int_{-1/2}^{1/2} 1\Big(\cos\omega x - \cancel{j\sin\omega x}\Big)\,dx && \text{expanding the complex exponential}\\
&= \int_{-1/2}^{1/2} \cos\omega x\,dx = \frac{\sin(\omega/2)}{\omega/2}. && \text{since sines are odd}
\end{aligned}
$$

We note that the transform of this function is real; this is because the given unit pulse is even, therefore the product of the pulse and $\sin\omega x$ is odd, and hence its integral is zero. In general, **even functions have real Fourier Transforms.**

11.2 Example 2.

We shall now find the Fourier Transform of a single sawtooth shape, $S(x)$, as shown below.



The Transform follows:

$$
\begin{aligned}
\mathcal{F}\left[S(x)\right] &= \int_{-\infty}^{\infty} S(x)e^{-j\omega x}\,dx && \text{by definition}\\
&= \int_{-\infty}^{-1} 0\,e^{-j\omega x}\,dx + \int_{-1}^{1} x\,e^{-j\omega x}\,dx + \int_{1}^{\infty} 0\,e^{-j\omega x}\,dx && \text{splitting into three regions}\\
&= \int_{-1}^{1} x\Big(\cos\omega x - j\sin\omega x\Big)\,dx && \text{expanding the complex exponential}\\
&= \int_{-1}^{1} \Big[\cancel{x\cos\omega x} - j\,x\sin\omega x\Big]\,dx && \text{The cCancelled integrand is odd}\\
&= -j\int_{-1}^{1} x\sin\omega x\,dx \quad = \quad 2j\left(\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right). && \text{using integration by parts}
\end{aligned}
$$

In this case the sawtooth function is odd and therefore its product with $\cos\omega x$ is also odd. Hence the integral of that component is zero. The consequence is that the Fourier Transform is purely imaginary, a result that is true for all odd functions.

We have the general result:

If $f(x)$ is even then $\mathcal{F}[f(x)]$ is real.

If $f(x)$ is odd then $\mathcal{F}[f(x)]$ is imaginary.

If $f(x)$ is neither even nor odd then the real part of $\mathcal{F}[f(x)]$ corresponds to the even component of $f(x)$ and the imaginary part corresponds to the odd component.

11.3 Example 3

Let us find the Fourier Transform of the unit impulse at $x = a$. We have,

$$\mathcal{F}[\delta(x-a)] \quad = \quad \int_{-\infty}^{\infty} \delta(x-a)e^{-j\omega x}\,dx \quad = \quad e^{-j\omega a}. \tag{251}$$

This integral has used the general result for integrals involving the unit impulse:

$$\int_{-\infty}^{\infty} \delta(x-a)g(x)\,dx = g(a), \tag{252}$$

i.e. that the integral is given by the rest of the integrand, $g(x)$, being evaluated where the impulse occurs.

We may play with this result a little. Given the above, then we have

$$\mathcal{F}[\delta(x+a)] \quad = \quad e^{j\omega a}. \tag{253}$$

This follows either by integrating this new transform from scratch, or else by replacing a by $-a$ in Eq. (251).

We can add these two results to get

$$\mathcal{F}[\delta(x+a)+\delta(x-a)] \quad = \quad e^{j\omega a}+e^{-j\omega a} \quad = \quad 2\cos\omega a, \tag{254}$$

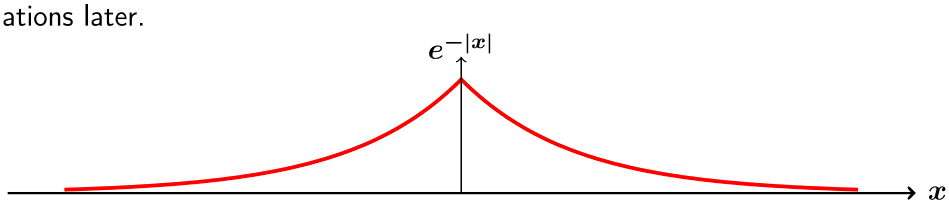
or we may subtract them to get,

$$\mathcal{F}[\delta(x+a)-\delta(x-a)] \quad = \quad e^{j\omega a}-e^{-j\omega a} \quad = \quad 2j\sin\omega a. \tag{255}$$

We will also use these results later.

11.4 Example 4

We will consider the Fourier Transform of $e^{-|x|}$. This seems like a bit of strange function to choose for it doesn't arise too often in engineering, but it too will prove useful when we solve some partial differential equations later.



First we notice that this function is even, and therefore we expect the Fourier Transform to be real. In what follows we will get to a point where I either have to use integration by parts or else to take the real part of a complex integral; I have decided to choose the latter route because that is my favoured way.

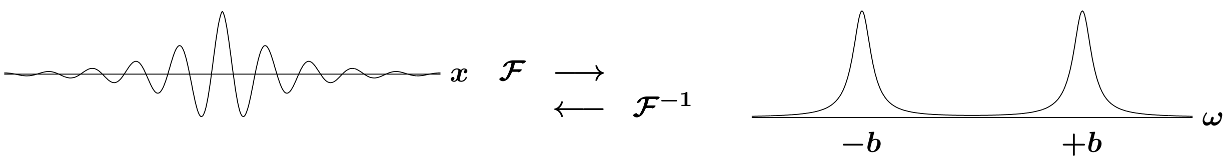
$$
\begin{aligned}
\mathcal{F}\left[e^{-|x|}\right] &= \int_{-\infty}^{\infty} e^{-|x|}e^{-j\omega x}\,dx && \text{by definition}\\
&= \int_{-\infty}^{\infty} \underbrace{e^{-|x|}}_{\text{even}}\,(\underbrace{\cos\omega x}_{\text{even}} - j\,\underbrace{\sin\omega x}_{\text{odd}})\,dx \\
&= \int_{-\infty}^{\infty} e^{-|x|}\cos\omega x\,dx && \text{but this integrand is even, so...}\\
&= 2\int_{0}^{\infty} e^{-|x|}\cos\omega x\,dx && \text{note the lower limit change}\\
&= 2\int_{0}^{\infty} e^{-x}\cos\omega x\,dx && e^{-|x|} = e^{-x} \text{ when } x \geq 0\\
&= 2\,\text{Real}\int_{0}^{\infty} e^{-x}e^{j\omega x}\,dx && \text{Added an imaginary term for convenience}\\
&= 2\,\text{Real}\int_{0}^{\infty} e^{-(1-j\omega)x}\,dx \\
&= 2\,\text{Real}\left[\frac{e^{-(1-j\omega)x}}{-(1-j\omega)}\right]_{0}^{\infty} \\
&= 2\,\text{Real}\left[\frac{1}{1-j\omega}\right] \\
&= 2\,\text{Real}\left[\frac{1+j\omega}{1+\omega^2}\right] \\
&= \frac{2}{1+\omega^2}.
\end{aligned}
$$

I have performed the above integration in quite pedantic detail. If you can speed that up then that would be excellent!

12 Physical meaning of the Fourier Transform

Having experienced four examples of Fourier Transforms, it is quite likely that the first thought is that these are merely integrals, and that the whole idea is that it is a mathematical trick. Well, it is a trick but it is also a meaningful trick. Here are some examples of Fourier Transform pairs. In all cases I have used even functions to transform because these have real transforms.

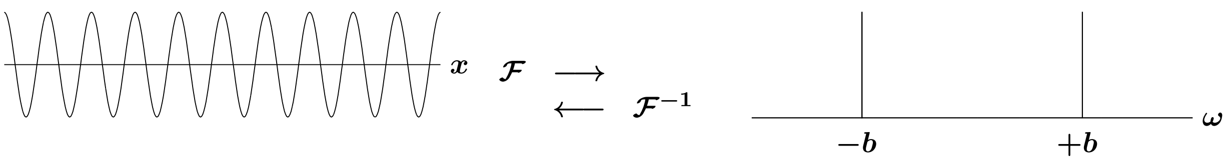
Example 5. $\mathcal{F}\left[e^{-a|x|}\cos bx\right] = \dfrac{a}{a^2+(b+\omega)^2} + \dfrac{a}{a^2+(b-\omega)^2}$



In this case the peaks of the Fourier Transform occur at $\omega = \pm b$. This means that the transform is telling us that the original signal, $e^{-a|x|}\cos bx$, contains a strong component with frequency, b, as is quite obvious!

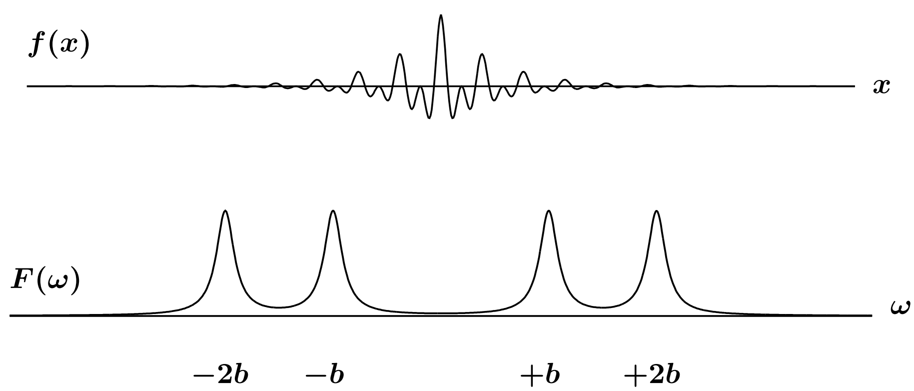
In the above, if we were to decrease the value of a, then the exponential decays more slowly than is depicted here. The consequence for the transform is that the two peaks will narrow and the maximum values ($\simeq 1/a$) will increase. In the limit as $a \to 0$ we obtain the following situation where the transform of $\cos bx$ is the sum of two delta functions.

Example 6. $\mathcal{F}[\cos bx] = \frac{1}{2}\Big[\delta(b+\omega)+\delta(b-\omega)\Big]$



Example 7. $\mathcal{F}\left[e^{-a|x|}\cos bx + e^{-a|x|}\cos 2bx\right] =$

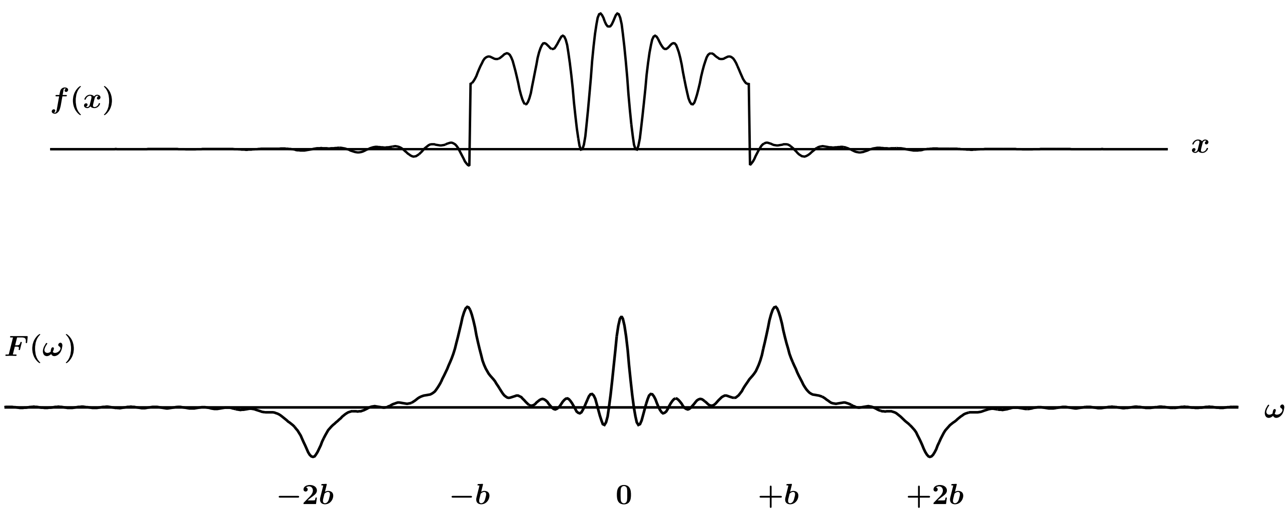
$$\frac{a}{a^2+(b+\omega)^2} + \frac{a}{a^2+(b-\omega)^2} + \frac{a}{a^2+(2b+\omega)^2} + \frac{a}{a^2+(2b-\omega)^2}$$



In this example we have found the transform of a sum of two functions and this is the same as the sum of the individual transforms. A nice result. Even nicer is the fact that the Fourier Transform clearly has well-defined peaks at $\omega = \pm b$ and $\pm 2b$. While these frequencies are seen clearly in the **red** and **blue** formulae making up $f(x)$, they are not so obvious in the graph of $f(x)$. It is this property that is used in CAT scans and other scientific applications.

Example 8. $\mathcal{F}\left[e^{-a|x|}\cos bx - 0.5e^{-a|x|}\cos 2bx + P(x)\right] =$

$$\frac{a}{a^2+(b+\omega)^2}+\frac{a}{a^2+(b-\omega)^2}+\frac{-0.5a}{a^2+(2b+\omega)^2}+\frac{-0.5a}{a^2+(2b-\omega)^2}+\frac{\sin(\omega/2)}{(\omega/2)}.$$



In the above the green colour represents both the unit pulse (see Example 3) and its Fourier Transform in their respective places.

The function, $f(x)$, now looks very strange but the Fourier Transform has uncovered its essential character. We can see the peaks at $\omega = \pm b$ and $\omega = \pm 2b$, although the amplitude of the latter is negative and has half the magnitude — this is consistent with the corresponding blue and red functions in $f(x)$. In addition, there is a peak at $\omega = 0$ and this corresponds to the unit pulse. Thus the transform provides us with a lot of information about the frequency content of the original signal.

13 Fourier Transforms of derivatives

The main aim of this part of the `ME20021` unit is the solution of PDEs using Fourier Transforms, and therefore we need to find out a few things about the transforms of derivatives.

13.1 The Fourier Transform of a single derivative

We shall find the Fourier Transform of $f'(x)$:

$$\begin{aligned}\mathcal{F}\left[f'(x)\right] &= \int_{-\infty}^{\infty} f'(x)\, e^{-j\omega x}\, dx && \text{we'll integrate the } f' \\ &= \underbrace{\cancel{\Big[f\Big]\Big[e^{-j\omega x}\Big]_{-\infty}^{\infty}}}_{f \to 0 \text{ as } |x|\to\infty} - \int_{-\infty}^{\infty}\Big[f\Big]\Big[-j\omega e^{-j\omega x}\Big]\, dx && \text{by parts once} \\ &= j\omega\int_{-\infty}^{\infty} f\, e^{-j\omega x}\, dx \quad = \quad j\omega F(\omega).\end{aligned}$$

The most important aspect of this analysis was the assumption that f tends to zero as x tends to $\pm\infty$. This was one of the sufficient conditions that were stated in §10.4.

Subject to having $f \to 0$ as $|x| \to \infty$, an x-derivative in the spatial domain is equivalent to multiplication by $j\omega$ in the frequency domain.

13.2 The Fourier Transform of a second derivative

We shall follow the same idea but will integrate by parts twice.

$$\begin{aligned}
\mathcal{F}\left[f''(x)\right] &= \int_{-\infty}^{\infty} f''(x)\, e^{-j\omega x}\, dx \\
&= \underbrace{\cancel{\left[f'\right]\left[e^{-j\omega x}\right]_{-\infty}^{\infty}}}_{f' \to 0 \text{ as } |x| \to \infty} - \underbrace{\cancel{\left[f\right]\left[-j\omega e^{-j\omega x}\right]_{-\infty}^{\infty}}}_{f \to 0 \text{ as } |x| \to \infty} + \int_{-\infty}^{\infty} \left[f\right]\left[j^2\omega^2 e^{-j\omega x}\right] dx \\
&= (j\omega)^2 \int_{-\infty}^{\infty} f\, e^{-j\omega x}\, dx \quad = \quad -\omega^2 F(\omega).
\end{aligned}$$

Thus we need both f and f' to tend to zero as $x \to \pm\infty$ for this result to be valid. The manner in which this analysis proceeds tells us that there is a simple rule for yet higher derivatives, namely that

$$\mathcal{F}\left[\frac{d^n f}{dx^n}\right] = (j\omega)^n F(\omega),$$

provided that f and its first $n-1$ derivatives tend to zero when $|x|$ is large.

14 Useful Theorems

As with Laplace Transforms there is a small set of useful theorems that may occasionally be used to assist in solving PDEs. Thus we have two shift theorems, a symmetry theorem and the convolution theorem. I will cover these in turn below.

14.1 The Shift Theorem in x

We shall start with defining $\mathcal{F}\left[f(x)\right] = F(\omega)$. Now we shall shift the origin in x and attempt to find the transform of that function in terms of $F(\omega)$.

$$\begin{aligned}
\mathcal{F}\left[f(x-a)\right] &= \int_{-\infty}^{\infty} f(x-a)\, e^{-j\omega x}\, dx && \text{By definition.} \\
& && x-a \text{ is awkward, so let } \xi = x-a \text{ hence } d\xi = dx. \\
&= \int_{-\infty}^{\infty} f(\xi) e^{-j\omega(\xi+a)}\, d\xi && \text{Limits unchanged.} \\
&= e^{-j\omega a} \underbrace{\int_{-\infty}^{\infty} f(\xi) e^{-j\omega\xi}\, d\xi}_{\mathcal{F}\left[f(x)\right]} \\
&= e^{-j\omega a} F(\omega).
\end{aligned}$$

An example of the use of this is the following. We know that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$, then the shift theorem tells us that the Fourier Transform of $e^{-|x-5|}$ is $2e^{-5j\omega}/(1+\omega^2)$.

14.2 The Shift Theorem in ω

Although the above shift theorem was quite quick to prove, this one is even faster!

$$\begin{aligned}
\mathcal{F}\left[f(x)\, e^{jax}\right] &= \int_{-\infty}^{\infty} f(x)\, e^{jax} e^{-j\omega x}\, dx && \text{By definition} \\
&= \int_{-\infty}^{\infty} f(x) e^{-j(\omega - a)x}\, dx \\
&= F(\omega - a). && \text{after noting that } \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\, dx = F(\omega)
\end{aligned}$$

An example of the use of this theorem is the following. Given that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$ then $\mathcal{F}\left[e^{-|x|+6jx}\right] = 2/(1+(\omega-6)^2)$.

14.3 The Symmetry Theorem

The similarity between the above two shift theorems (namely that an origin shift in either x or ω is equivalent to multiplication by a complex exponential in either ω or x) is not an accident, but is based on the very great similarity between the definitions of the Fourier Transform and of its inverse given in Eqs. (249) and (250). This similarity motivates questions like the following:

Can we use the fact that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$, which is a fairly straightforward integral to perform, to help us find $\mathcal{F}\left[2/(1+x^2)\right]$, which is very considerably more difficult to evaluate?

The answer is yes; here is the Symmetry Theorem:

$$\text{If} \quad \mathcal{F}[f(x)] = F(\omega) \quad \text{then} \quad \mathcal{F}[F(x)] = 2\pi f(-\omega). \tag{256}$$

The proof of this may be found in the Appendix 1 at the end of these notes, but this is not examinable.

14.3.1 Example 9.

We shall answer the question in red which motivated the theorem. If we let $f(x) = e^{-|x|}$, then $F(\omega) = 2/(1+\omega^2)$. Hence the theorem gives us

$$\mathcal{F}\Big[\underbrace{\frac{2}{1+x^2}}_{F(x)}\Big] = 2\pi \underbrace{e^{-|-\omega|}}_{f(-\omega)} = 2\pi e^{-|\omega|}. \tag{257}$$

14.3.2 Example 10.

The aim is to find the Fourier Transform of 1, which is a case that is not guaranteed to have a transform. If we let $f(x) = \delta(x)$, then $F(\omega) = 1$ (see Example 3 with $a = 0$). Then the Symmetry Theorem states that,

$$\mathcal{F}[\underbrace{\ 1\ }_{F(x)}] = 2\pi \underbrace{\delta(-\omega)}_{f(-\omega)} = 2\pi\delta(\omega).$$

Physically, this result means that all of the frequency content of a constant signal is concentrated at $\omega = 0$.

14.3.3 Example 11.

We have already seen in Example 3 (specifically Eq. (254)) that,

$$\mathcal{F}\left[\delta(x+a)+\delta(x-a)\right]=2\cos a\omega.$$

If we divide both sides by 2 then we have

$$\mathcal{F}\left[\frac{\delta(x+a)+\delta(x-a)}{2}\right]=\cos a\omega.$$

If we define

$$f(x)=\Big[\delta(x+a)+\delta(x-a)\Big]/2$$

and

$$F(\omega)=\cos a\omega,$$

then the Symmetry Theorem tells us that,

$$\mathcal{F}\,[\underbrace{\cos ax}_{F(x)}]=2\pi\times\underbrace{\tfrac{1}{2}\Big[\delta(-\omega+a)+\delta(-\omega-a)\Big]}_{f(-\omega)}=\pi\Big[\delta(\omega-a)+\delta(\omega+a)\Big],$$

since $\delta(-b)=\delta(b)$.

14.4 The Convolution Theorem

This operates in the same sort way as for the Laplace Transform except that the convolution integral itself has different limits. For the Fourier Transform, the convolution of two functions, $f(x)$ and $g(x)$ is defined to be,

$$f*g \quad = \quad \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \quad = \quad \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi.$$

The Laplace Transform version has $\xi=0$ and $\xi=x$ as its limits.

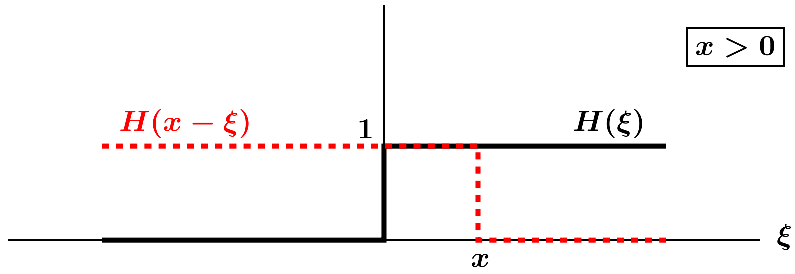
For the Fourier Transform, the Convolution Theorem is

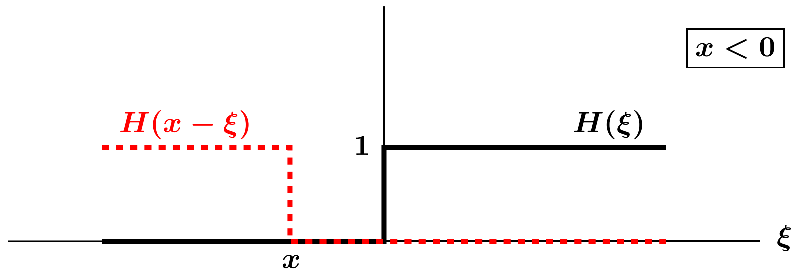
$$\mathcal{F}\,[f*g]=F(\omega)G(\omega),$$

where $\mathcal{F}\,[f(x)]=F(\omega)$ and $\mathcal{F}\,[g(x)]=G(\omega)$. So **the transform of the convolution is equal to the product of the transforms.** The proof of this is contained in Appendix 2 at the end of these notes, and this too is very definitely not needed for the exams.

14.4.1 A convolution of unit step functions

We shall find the convolution of the unit step function, $H(x)$, with itself: $H(x)*H(x)$. The chief difficulty with this example is with the manipulation of the limits of integration. It is best to sketch a couple of diagrams first.





The convolution is given by

$$\int_{-\infty}^{\infty} H(\xi)H(x-\xi)\,d\xi.$$

We may use the above diagrams to evaluate this integral. The integrand is given by the product of the values of the black line and those of the red line. For the first case, $x > 0$, the product of the step functions is equal to 1 in the range $0 < \xi < x$ and is zero otherwise. For the second case the product is equal to zero everywhere. Hence,

$$\int_{-\infty}^{\infty} H(\xi)H(x-\xi)\,d\xi = \left\{ \begin{array}{ll} \displaystyle\int_0^x 1\,d\xi = x & (x > 0), \\ 0 & (x < 0) \end{array} \right\} = xH(x).$$

15 An ODE example

As a final preparation for solving PDEs, we shall solve the following ODE,

$$\frac{dy}{dx} + y = \left\{ \begin{array}{ll} e^{-2x} & (x > 0), \\ 0 & (x < 0) \end{array} \right\} = e^{-2x}H(x).$$

We solved this in ME10305 using the Particular Integral and Complementary Function approach, and also using Laplace Transforms. Admittedly, the above equation looks odd especially since I have given no initial condition. One could assume that $y = 0$ is an initial condition at a negatively infinite value of x. Alternatively one can see that the Complementary Function part of the solution is e^{-x}, which decays exponentially, and therefore any nonzero initial condition can be imposed at such a large but negative value of x that y will have decayed virtually to zero by the time $x = 0$ is reached which is when the right hand side suddenly becomes nonzero. However, my intention with this is that $y = 0$ is assumed when $x < 0$ and that the above problem is essentially the same as the problem, $y' + y = e^{-2x}$ subject to $y(0) = 0$.

We start by taking the Fourier Transform of the ODE. If we define $Y(\omega) = \mathcal{F}[y(x)]$ then §13.1 gives $\mathcal{F}[y'] = j\omega Y$. The transform of the right hand side of the ODE is

$$\begin{aligned}
\mathcal{F}\left[e^{-2x}H(x)\right] &= \int_{-\infty}^{\infty} e^{-2x}H(x)e^{-j\omega x}\,dx && \text{By definition} \\
&= \int_0^{\infty} e^{-2x} \times 1 \times e^{-j\omega x}\,dx && H = 0 \text{ when } x < 0 \\
&= \int_0^{\infty} e^{-(2+j\omega)x}\,dx \\
&= \frac{1}{2+j\omega}.
\end{aligned}$$

Now we can transform the given equation:

$$\begin{aligned}
(\omega j)Y + Y &= \frac{1}{2+\omega j} \\
\implies \quad (1+\omega j)Y &= \frac{1}{2+\omega j} \\
\implies \quad Y &= \frac{1}{(1+\omega j)(2+\omega j)} \\
\implies \quad Y &= \frac{1}{1+\omega j} - \frac{1}{2+\omega j} && \text{partial fractions}
\end{aligned}$$

Hence

$$y = \left(e^{-x} - e^{-2x}\right)H(x).$$

This completes the preparation and background on Fourier Transforms. The rest of the Fourier Transform lecture material consists solely of solving PDEs.

16 Fourier Transform solutions of PDEs

16.1 Solutions of Fourier's equation

We will solve Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \tag{258}$$

subject to the initial condition that

$$\theta = f(x) \qquad \text{at } t = 0. \tag{259}$$

We have to assume that the initial temperature profile, $f(x)$, must decay to zero as $x \to \pm\infty$ in order to be able to use Fourier Transforms. The solution procedure follows three steps:

(i) take the Fourier Transform of the given equation,

(ii) solve the transformed equation, an ODE,

(iii) take the inverse Fourier Transform to find the desired solution.

Note: In many cases the final answer has to be written in terms of an integral, but sometimes one may be able to find an explicit expression for the solution.

First we let $\Theta(\omega, t)$ be the Fourier Transform of $\theta(x, t)$ with respect to x. Thus,

$$\Theta(\omega, t) = \mathcal{F}\left[\theta(x, t)\right] = \int_{-\infty}^{\infty} \theta\, e^{-j\omega x}\, dx. \tag{260}$$

Now we'll take the Fourier Transform of the second derivative:

$$\begin{aligned}
\mathcal{F}\Big[\frac{\partial^2\theta}{\partial x^2}\Big] &= \int_{-\infty}^{\infty} \frac{\partial^2\theta}{\partial x^2} e^{-j\omega x}\, dx \\
&= \Big[\frac{\partial\theta}{\partial x}\Big]\Big[e^{-j\omega x}\Big]_{-\infty}^{\infty} - \Big[\theta\Big]\Big[-j\omega e^{-j\omega x}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \Big[\theta\Big]\Big[-\omega^2 e^{-j\omega x}\Big] \\
&= -\omega^2\Theta.
\end{aligned} \tag{261}$$

Note that we have used the facts that both θ and its x-derivative tend to zero as $x \to \pm\infty$.

The Fourier Transform of the time-derivative term is relatively straightforward:

$$\begin{aligned}
\mathcal{F}\Big[\frac{\partial\theta}{\partial t}\Big] &= \int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t}\, e^{-j\omega x}\, dx \\
&= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \theta\, e^{-j\omega x}\, dx \\
&= \frac{\partial\Theta}{\partial t}.
\end{aligned} \tag{262}$$

Note that we have effectively swapped the order of differentiation with respect to t and integration with respect to x when deriving this result.

Now Fourier's equation, (258), is transformed to

$$\frac{\partial\Theta}{\partial t} = -\alpha\omega^2\Theta. \tag{263}$$

This ordinary differential equation has the solution,

$$\Theta = A(\omega)e^{-\alpha\omega^2 t}, \tag{264}$$

where the 'constant of integration' may be treated as a function of ω. While this treatment might seem strange, the ordinary differential equation has t as its independent variable, and therefore ω is simply a passive parameter. Therefore it makes sense to ensure that A is as general as possible by allowing it to vary with ω.

The constant of integration may be found by applying an appropriate initial condition, but the only one which is available at present is Eq. (259), for θ. However, we may take the Fourier Transform of this to obtain,

$$\Theta = \mathcal{F}[f(x)] = F(\omega) \qquad \text{at } t = 0. \tag{265}$$

Therefore the setting of $t = 0$ in Eq. (264) yields,

$$A = F, \tag{266}$$

and so the Fourier Transform of the desired solution is,

$$\Theta = F(\omega)e^{-\alpha\omega^2 t}. \tag{267}$$

We may now apply the formula for the inverse Fourier Transform (see Eq. (250)) to obtain the final solution,

$$\boxed{\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha\omega^2 t}e^{j\omega x}\,d\omega.} \tag{268}$$

In general it is difficult to simplify this final integral in any meaningful way, and therefore we have to leave it as it is. Clearly, for any chosen pair of values of x and t the integral may be evaluated numerically with ease and to any desired accuracy. If a sufficient number of pairs of x and t are used it becomes possible to obtain either a contour plot of the evolution of θ in space and time, or else to show how the temperature profile varies with time.

16.2 Solution of Laplace's equation

We now solve,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0, \tag{269}$$

in the ranges $-\infty < x < \infty$ and $0 \le y < \infty$. The boundary conditions are that,

$$\theta = f(x) \quad \text{on } y = 0 \qquad \text{and} \qquad \theta \to 0 \quad \text{as } y \to \infty. \tag{270}$$

This system represents steady two-dimensional conduction in a half-plane where the temperature profile on the edge is $f(x)$.

Once more it is assumed that $\theta \to 0$ as $x \to \pm\infty$ so that it is possible to take Fourier Transforms with respect to x. We now use $\Theta(\omega, y)$ to denote the Fourier Transform of $\theta(x, y)$ with respect to x. Given our experience for Fourier's equation, it is clear that Laplace's equation transforms to

$$\frac{\partial^2\Theta}{\partial y^2} - \omega^2\Theta = 0. \tag{271}$$

The solution of this equation may be written as,

$$\Theta = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}. \tag{272}$$

We may now apply the large-y boundary condition, namely that $\theta \to 0$ or, equivalently, that $\Theta \to 0$. Although this might seem straightforward to do (and most people would immediately say that $A = 0$ in order to remove the exponentially growing solution) it is not quite as straightforward as one might think. The reason is that ω may also take negative values; see the definition of the Inverse Fourier Transform given in Eq. (250) where $-\infty < \omega < \infty$. Therefore we have the rather unusual reasoning:

$$\text{When } \omega > 0 \text{ we need } A = 0 \text{ so that } \Theta = B(\omega)e^{-\omega y}$$
$$\text{When } \omega < 0 \text{ we need } B = 0 \text{ so that } \Theta = A(\omega)e^{\omega y}.$$

The good news is that we may combine these two cases into one formula in the following way. Let,

$$\Theta = C(\omega)e^{-|\omega|y}, \tag{273}$$

where C is the arbitrary constant. We may now apply the boundary condition at $y = 0$, namely that $\theta = f(x)$ or that $\Theta = F(\omega)$. Hence $C = F$, and we obtain the transformed solution,

$$\Theta(\omega, y) = F(\omega)\, e^{-|\omega|y}. \tag{274}$$

It is easy to write down the inverse Fourier Transform of this:

$$\theta(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{-|\omega|y}\, e^{j\omega x}\, d\omega. \tag{275}$$

Now Eq. (274) is the product of two functions of ω, and therefore it is possible, in principle, to apply the Convolution Theorem to find its inverse transform. Earlier in the notes (see Eq. (257)) we used the Symmetry Theorem to show that,

$$\mathcal{F}\Big[\frac{2}{1+x^2}\Big] = 2\pi e^{-|\omega|}.$$

In a similar way it is also possible to show that,

$$\mathcal{F}\Big[\frac{2a}{a^2+x^2}\Big] = 2\pi e^{-|\omega|a}, \tag{276}$$

where a is a constant. For a Fourier Transform with respect to x, we might use y as an alternative constant to a to obtain,

$$\mathcal{F}\Big[\frac{2y}{y^2+x^2}\Big] = 2\pi e^{-|\omega|y}, \tag{277}$$

and hence,

$$\mathcal{F}\Big[\frac{1}{\pi} \times \frac{y}{y^2+x^2}\Big] = e^{-|\omega|y}. \tag{278}$$

So the application of the Convolution Theorem to Eq. (274) yields,

$$\begin{aligned} \theta &= f(x) * \Big[\frac{1}{\pi}\ \frac{y}{y^2+x^2}\Big] \\ &= \int_{-\infty}^{\infty} f(\xi - x)\Big[\frac{1}{\pi} \times \frac{y}{y^2+\xi^2}\Big]\, d\xi. \end{aligned} \tag{279}$$

Clearly this provides an alternative integral expression for the final solution to the one in Eq. (275).

Equation (279) may be used very easily if the initial condition is a unit impulse, $f(x) = \delta(x)$. The integral in (279) now becomes,

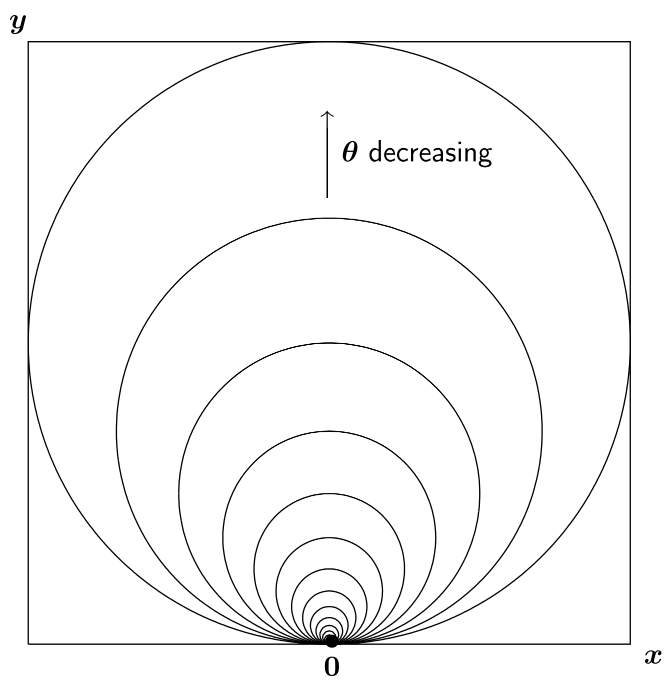
$$\theta = \int_{-\infty}^{\infty} \delta(\xi - x)\Big[\frac{1}{\pi} \times \frac{y}{y^2+\xi^2}\Big]\, d\xi = \frac{1}{\pi} \times \frac{y}{y^2+x^2}. \tag{280}$$

Of course, it is worth checking whether this simple-looking solution is reasonable. On $y = 0$ the function is clearly zero, which is consistent with the delta function boundary condition, except perhaps at the origin itself. As we move away from the origin along a straight line (where we may set $x = r\cos\theta$ and $y = r\sin\theta$ and therefore we are moving along the line θ =constant and $r \to \infty$), then θ is inversely proportional to the distance from the origin, i.e. it decays, which is also what we expect. Finally, if we set $x = 0$, then $\theta = 1/y$, which becomes infinite as we approach the origin, which is where we have sited the unit impulse. Indeed, we also get an infinite limit for all lines of approach to the origin, except on $y = 0$. This, unfortunately, is one of the many odd things that can happen with a unit impulse.

Contours of $\theta =$ constant are circular arcs, as may be seen by the following analysis. If we substitute $\theta = 1/(2\pi\alpha)$ into Eq. (280), where α is a constant, and where 2π is there for pure numerical convenience, then we obtain,

$$\frac{y}{\pi(x^2+y^2)} = \frac{1}{2\pi\alpha} \qquad \Rightarrow \qquad x^2 + y^2 = 2\alpha y \qquad \Rightarrow \qquad x^2 + (y-\alpha)^2 = \alpha^2. \quad (281)$$

This is the equation of a circle of radius α which is centred at $x = 0$ and $y = \alpha$. So, bizarrely, and again because of the presence of the unit impulse, all of these circular contours pass through the origin, as may be seen in the following Figure.



Depicting the contours (i.e. isotherms) of the solution given by Eq. (280). The x-axis is horizontal, and the unit impulse is placed at the origin, which is the point through which all the circular contours pass.

16.3 Fourier Sine and Cosine Transforms.

When solving partial differential equations in an infinite domain (i.e. in $-\infty < x < \infty$), it is often necessary to use the Fourier Transform. However, not all problems are defined on an infinite domain, but some are defined on a semi–infinite domain ($0 \le x < \infty$). This where we need to use either the Fourier Sine Transform (FST) or the Fourier Cosine Transform (FCT).

The FCT and the FST are intimately related to the Fourier Transform and they and their inverses may be derived from it and its inverse. The Fourier Cosine Transform pair is given by

$$F_c(\omega) = \mathcal{F}_c[f(x)] = \int_0^\infty f(x)\ \cos\omega x\,dx \tag{282}$$

and

$$f(x) = \mathcal{F}_c^{-1}[F_c(\omega)] = \frac{2}{\pi}\int_0^\infty F_c(\omega)\ \cos\omega x\,d\omega. \tag{283}$$

The Fourier Sine Transform pair is given by

$$F_s(\omega) = \mathcal{F}_s[f(x)] = \int_0^\infty f(x)\ \sin\omega x\,dx \tag{284}$$

and

$$f(x) = \mathcal{F}_s^{-1}[F_s(\omega)] = \frac{2}{\pi}\int_0^\infty F_s(\omega)\ \sin\omega x\,d\omega. \tag{285}$$

If we are solving an equation for $u(x,y)$, say, where transforms are being taken in the x-direction, then the FST is used when $u(x,y)$ is given on the boundary $x=0$, and that the FCT is used when the first x-derivative of $u(x,y)$ is given on $x=0$. The reasons for this are technical, and they arise naturally during the integration by parts process for evaluating the transforms of derivatives.

The similarity between the formulae for the Fourier Sine transform and its inverse, and between the Fourier Cosine Transform and its inverse is even stronger than between the Fourier Transform and its inverse, as given in Eqs. (249) and (250). The reason I have mentioned this is that the equivalent Symmetry Theorems for the sine and cosine transforms become almost trivial to write down. Thus if $\mathcal{F}_c[f(x)] = F_c(\omega)$ then $\mathcal{F}_c[F_c(t)] = \frac{1}{2}\pi f(\omega)$, with a similar-looking formula for the sine transform. So, for example, it is straightforward to show that,

$$\mathcal{F}_c[e^{-x}] = \int_0^\infty e^{-x}\ \cos\omega x\,dx = \frac{1}{1+\omega^2}, \tag{286}$$

and therefore the Symmetry Theorem then tells us that,

$$\mathcal{F}_c\Big[\frac{1}{1+x^2}\Big] = \tfrac{1}{2}\pi e^{-\omega}. \tag{287}$$

16.4 Some Fourier Sine Transform examples.

16.4.1 Example 1

We will consider the following unsteady one-dimensional heat transfer problem. A semi-infinite solid, which occupies the region, $0 \le x < \infty$, has the temperature profile, $\theta = f(x)$, at $t=0$. However, the $x=0$ end of this region is maintained at the temperature, $\theta = 0$. Determine the evolution of the temperature profile.

Let $\Theta_s(\omega,t)$ be the Fourier Sine Transform of $\theta(x,t)$ with respect to x, i.e. that

$$\Theta_s = \mathcal{F}_s[\theta] = \int_0^\infty \theta\sin\omega x\,dx.$$

We will be solving Fourier's equation, as given by Eq. (258), and therefore we will need to take the Fourier Sine Transform of this equation. Beginning with the time-derivative term, we get

$$\mathcal{F}_s\Big[\frac{\partial\theta}{\partial t}\Big] = \int_0^\infty \frac{\partial\theta}{\partial t}\sin\omega x\,dx = \frac{\partial}{\partial t}\int_0^\infty \theta\,\sin\omega x\,dx = \frac{\partial\Theta_s}{\partial t}. \tag{288}$$

For the term with the second derivative in x, we have

$$\begin{aligned}\mathcal{F}_s\Big[\frac{\partial^2\theta}{\partial x^2}\Big] &= \int_0^\infty \frac{\partial^2\theta}{\partial x^2}\sin\omega x\,dx \\ &= \Big[\frac{\partial\theta}{\partial x}\Big]\Big[\sin\omega x\Big]_0^\infty - \Big[\theta\Big]\Big[\omega\cos\omega x\Big]_0^\infty + \int_0^\infty \Big[\theta\Big]\Big[-\omega^2\sin\omega x\Big]\,dx \\ &= -\omega^2\Theta_s.\end{aligned} \tag{289}$$

Note that we had to use the "$\theta = 0$ at $x = 0$" boundary condition in the **blue terms** in Eq. (289) in order to derive the final line of (289). Note also that, if the boundary condition at $x = 0$ had been a Neumann condition (the gradient of θ is specified) then we would not be able to proceed further because the value of θ at $x = 0$ is needed there. Thus the Fourier Sine Transform **must** be used with Dirichlet boundary conditions.

We also assumed that θ and its derivatives decay to zero as $x \to \infty$. Therefore Fourier's equation transforms to,

$$\frac{\partial\Theta_s}{\partial t} = -\alpha\omega^2\Theta_s, \tag{290}$$

and the solution is,

$$\Theta_s = A(\omega)e^{-\alpha\omega^2 t}, \tag{291}$$

where $A(\omega)$ is the *arbitrary* constant (from the point of view of t). The given initial condition is that $\theta = f(x)$ when $t = 0$; on taking the Fourier Sine Transform of this, we obtain the transformed version, namely,

$$\Theta_s = \mathcal{F}_s[f(x)] = F_s(\omega) \qquad \text{when } t = 0. \tag{292}$$

Substitution of Eq. (292) into Eq. (291) yields $A = F_s$, and hence

$$\Theta_s = F_s(\omega)e^{-\alpha\omega^2 t}. \tag{293}$$

The final solution is obtained by taking the inverse Fourier Sine Transform of this; we get,

$$\theta = \mathcal{F}_s{}^{-1}[\Theta_s] = \frac{2}{\pi}\int_0^\infty F_s(\omega)e^{-\alpha\omega^2 t}\sin\omega x\,d\omega. \tag{294}$$

16.4.2 Example 2

We will again solve Fourier's equation, but this time the initial temperature profile is $\theta = 0$, and the boundary at $x = 0$ is held at the temperature $\theta = 1$. This is an example of a sudden heating problem, the final solution representing the manner in which heat diffuses into a formerly cold domain.

Proceeding as for Example 1, we again get to the stage represented by Eq. (289), but now we need to use the fact that $\theta = 1$ at $x = 0$. Therefore the present equivalent of Eq. (289) is,

$$\mathcal{F}_s\Big[\frac{\partial^2\theta}{\partial x^2}\Big] = \omega - \omega^2\Theta_s. \tag{295}$$

In this case, Fourier's equation transforms to,

$$\frac{\partial\Theta_s}{\partial t} = \alpha\Big[\omega - \omega^2\Theta_s\Big], \tag{296}$$

and the solution is,

$$\Theta_s = \frac{1}{\omega} + A(\omega)e^{-\alpha\omega^2 t}, \tag{297}$$

using the Complementary Function / Particular Integral approach.

At $t = 0$ we have $\theta = 0$, and hence $\Theta_s = 0$ too. Subsitution of this fact into Eq. (297) gives $A = -1/\omega$, and therefore,

$$\Theta_s = \frac{1 - e^{-\alpha\omega^2 t}}{\omega}. \tag{298}$$

The inverse Fourier Sine Transform yields,

$$\theta = \frac{2}{\pi}\int_0^\infty \frac{1 - e^{-\alpha\omega^2 t}}{\omega}\sin\omega x\, d\omega. \tag{299}$$

As a final note on this example, there is another way in which this problem might be solved and it will be met in one of the units concerned with Heat Transfer. For now I will merely quote the solution:

$$\theta = \frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{\alpha t}}^\infty e^{-\xi^2}\, d\xi = \text{erfc}\Big[\frac{x}{2\sqrt{\alpha t}}\Big]. \tag{300}$$

This function is known as the complementary error function; you'll meet it formally one day.

16.5 A Fourier Cosine Transform example.

16.5.1 Example 3

This final example is a reprise of Example 1, above, but now we shall assume that the $x = 0$ boundary is insulated, i.e. that $\partial\theta/\partial x = 0$ there.

Let $\Theta_c(\omega, t)$ be the Fourier Cosine Transform of $\theta(x, t)$ with respect to x, i.e. that

$$\Theta_c = \mathcal{F}_c[\theta] = \int_0^\infty \theta\cos\omega x\, dx.$$

Again, we will be solving Fourier's equation as given by Eq. (258), and we will take the Fourier Cosine Transform of this equation. Beginning with the time-derivative term once more, we get

$$\mathcal{F}_c\Big[\frac{\partial\theta}{\partial t}\Big] = \int_0^\infty \frac{\partial\theta}{\partial t}\cos\omega x\, dx = \frac{\partial}{\partial t}\int_0^\infty \theta\,\cos\omega x\, dx = \frac{\partial\Theta_s}{\partial t}. \tag{301}$$

For the term with the second derivative in x, we have

$$\begin{aligned}
\mathcal{F}_c\left[\frac{\partial^2\theta}{\partial x^2}\right] &= \int_0^\infty \frac{\partial^2\theta}{\partial x^2}\cos\omega x\,dx \\
&= \left[\frac{\partial\theta}{\partial x}\right]\Big[\cos\omega x\Big]_0^\infty - \Big[\theta\Big]\Big[-\omega\sin\omega x\Big]_0^\infty + \int_0^\infty \Big[\theta\Big]\Big[-\omega^2\cos\omega x\Big]\,dx \\
&= -\omega^2\Theta_c.
\end{aligned} \tag{302}$$

Note that we had to use the "$\partial\theta/\partial x = 0$ at $x = 0$" boundary condition in the **red term** in this equation in order to derive the final result. Note also that, if the boundary condition at $x = 0$ had been a Dirichlet condition (where θ is specified) then we would not be able to proceed further because the value of $\partial\theta/\partial x$ at $x = 0$ is needed there. Thus the Fourier Cosine Transform must be used with Neumann boundary conditions.

Fourier's equation transforms to,

$$\frac{\partial\Theta_c}{\partial t} = -\alpha\omega^2\Theta_c, \tag{303}$$

and the solution is,

$$\Theta_c = A(\omega)e^{-\alpha\omega^2 t}. \tag{304}$$

The given initial condition is that $\theta = f(x)$ when $t = 0$; on taking the Fourier Cosine Transform of this, we obtain the transformed version, namely,

$$\Theta_c = \mathcal{F}_c[f(x)] = F_c(\omega) \qquad \text{when } t = 0. \tag{305}$$

Substitution of Eq. (305) into (304) yields $A = F_s$, and hence

$$\Theta_c = F_c(\omega)e^{-\alpha\omega^2 t}. \tag{306}$$

The final solution is obtained by taking the inverse Fourier Cosine Transform of this; we get,

$$\theta = \mathcal{F}_c^{-1}[\Theta_c] = \frac{2}{\pi}\int_0^\infty F_c(\omega)e^{-\alpha\omega^2 t}\cos\omega x\,d\omega. \tag{307}$$

Note: The analysis for this example is almost identical to that for Example 1. Apart from small details in the Integration by Parts, my general policy when teaching in a lecture theatre would be to retain Example 1 on the board somehow, then alter all the s-subscripts to c-subscripts, with a small adjustment in the integration-by-parts. That's it — no more than that! This is how similar these two problems are.

16.6 A final comment

The last three examples have only involved solutions of Fourier's equation. It is also possible to use the Fourier Sine and Cosine Transforms to solve Laplace's equation and the wave equation, but these feature in the final Problem Sheet and fairly detailed solutions are provided for these.

Appendix 1. A proof of the Symmetry Theorem

NOTE that the following derivation is for interest only.

By definition the Fourier Transform of $f(x)$ is given by,

$$\mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx = F(\omega), \tag{308}$$

and the corresponding inverse Fourier Transform is given by

$$\mathcal{F}^{-1}\left[F(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}\,d\omega = f(x). \tag{309}$$

The aim here is to find

$$\mathcal{F}\left[F(x)\right] = \int_{-\infty}^{\infty} F(x)e^{-j\omega x}\,dx. \tag{310}$$

It is important to notice that the argument to the exponential in (310) has the opposite sign to what we have in (309) and that the variables, x and ω, are in the 'wrong' places. These problems can be overcome by a series of shifty manoeuvres.

The first thing we must do is to interchange the roles of x and ω; this is realised by first changing the (dummy) variable of integration in Eq. (309) from ω to s to get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{jsx}\,ds.$$

Now we rewrite this expression in terms of ω instead of x:

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{js\omega}\,ds,$$

and then change the dummy variable from s to x:

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{j\omega x}\,dt.$$

Finally we replace every occurrence of ω by $-\omega$:

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{-j\omega x}\,dx.$$

Minor manipulations lead us to the Symmetry Theorem,

$$\mathcal{F}\left[F(x)\right] = \int_{-\infty}^{\infty} F(x)e^{-j\omega x}\,dx = 2\pi f(-\omega). \tag{311}$$

Appendix 2. A proof of the Convolution Theorem.

NOTE that the following derivation is for interest only, but the result is applied very frequently when solving ODEs and PDEs.

If $f(x)$ and $g(x)$ are given functions, both having Fourier transforms, then the **convolution of $f(x)$ and $g(x)$** is defined to be

$$f * g = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \quad \text{or} \quad \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi, \tag{312}$$

and we are interested in the Fourier transform of this function. **Note** that this is a different definition from that which is used in Laplace Transforms, where the integral runs from 0 to x only.

By definition,

$$\begin{aligned} \mathcal{F}[f*g] = & \int_{-\infty}^{\infty} e^{-j\omega x}\Big[\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi\Big]\,dx \\ = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega x} f(\xi)g(x-\xi)\,d\xi\,dx \\ = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega(x-\xi)}e^{-j\omega\xi} f(\xi)g(x-\xi)\,dt\,d\xi, \end{aligned} \tag{313}$$

where the last line involved changing the order of integration and a sneaky manipulation of the exponents in the exponentials. Now change variables from x to s according to $s = x - \xi$ $(ds = dx)$ and hence

$$\begin{aligned} \mathcal{F}[f*g] = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s)e^{-j\omega s} f(\xi)e^{-j\omega\xi}\,ds\,d\xi, \\ = & \Big[\int_{-\infty}^{\infty} g(s)e^{-j\omega s}\,ds\Big]\Big[\int_{-\infty}^{\infty} f(\xi)e^{-j\omega\xi}\,d\xi\Big] \\ = & G(\omega)F(\omega). \end{aligned} \tag{314}$$

The Fourier transform of the convolution of two functions is, therefore, the product of their respective transforms.

D.A.S.R. 14/02/2022

Department of Mechanical Engineering, University of Bath

Modelling Techniques S2 ME20021 Sheet 1

Separation of Variables for PDEs: Fundamental Solutions

These questions are designed to give you practice in finding suitable fundamental solutions of the various PDEs that we will be covering in my part of this unit. The questions are incomplete in the sense that the application of the very last boundary/initial condition is missing which will require the application of a Fourier Series formula. Later problem sheets will involve Fourier Series.

Q1. The aim is to solve Laplace's equation in an infinite strip.

(i) Find the particular solution of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the domain $0 \le x \le 1$, $0 \le y \le \infty$, which satisfies $u = 0$ on $x = 0$ and $x = 1$ and which decays as $y \to \infty$. Hence write down a Fourier Sine Series solution by superposing all such solutions obtained for positive integer values of n. If $u = f(x)$ at $y = 0$, then find a Fourier Sine Series expression for $f(x)$. [Note: this follows the lecture notes precisely, but do try it out first without reference to the notes.]

(ii) How does the solution found in part (i) need to change if one were to solve Laplace's equation in the domain, $0 \le x \le d$, $0 \le y \le \infty$, where $u = 0$ on both $x = 0$ and $x = d$?

(iii) [A bit awkward.] How does this solution change if the domain is now $-d \le x \le d$ with $u = 0$ on both of these boundaries?

Q2. Solve Fourier's equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

in the domain $0 \le x \le 1$, $0 \le t \le \infty$, which satisfies the boundary conditions $u = 0$ at both $x = 0$ and $x = 1$. Hence write down a Fourier Sine Series solution by superposing all such solutions. If $u = f(x)$ at $t = 0$, then find a Fourier Sine Series expression for $f(x)$.

Determine how the above changes when the spatial domain is $0 \le x \le d$ and when $-d \le x \le d$, as in Q1.

Q3. Solve the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions, $u = 0$ on $x = 0$ and $x = 1$, and write down the most general possible solution in terms of a Fourier Sine series.

(i) Apply the initial conditions, $u = f(x)$ and $\partial u/\partial t = 0$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series.

(ii) Apply the initial conditions, $u = 0$ and $\partial u/\partial t = f(x)$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series. What is the physical interpretation of this pair of initial conditions?

Q4. [Possibly the most difficult examinable PDE that I could set.] Vibrations of a taut string are often slightly damped, and therefore an extra damping term will be present in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Solve this equation in the domain $0 \le x \le 1$, $0 \le t \le \infty$, where u satisfies the boundary conditions $u = 0$ at both $x = 0$ and $x = 1$. Hence write down a Fourier Sine Series solution by superposing all such solutions.

[Note that we are assuming light damping which, in the present context, means that $k < \pi c$. You will find it easier to write things down by introducing the quantity, d_n, where $d_n = \sqrt{n^2c^2\pi^2 - k^2}$.]

Q5. The aim is to solve the wave equation for beams,

$$\frac{\partial^2 u}{\partial t^2} + c^4\frac{\partial^4 u}{\partial x^4} = 0,$$

which is used for relatively thick vibrating structures. Analytical progress may be made with pin-jointed beams for which the boundary conditions are that both $u = 0$ and $\partial^2 u/\partial x^2 =$ on $x = 0$ and $x = 1$.

Write down the most general possible solution in terms of a Fourier Sine series. Apply the initial conditions, $u = f(x)$ and $\partial u/\partial t = 0$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series.

Q6. Solve Fourier's equation,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}.$$

in the domain $0 \le x \le 1$, $0 \le t \le \infty$, which satisfies the boundary conditions $\partial u/\partial x = 0$ at both $x = 0$ and $x = 1$. (Note that sines do not satisfy these boundary conditions, and therefore you'll need to use something else....)

Department of Mechanical Engineering, University of Bath

S2 Modelling Techniques   Sheet 2

Separation of Variables for PDEs and Fourier Series

Q1. The steady temperature distribution $T(x, y)$ inside the infinitely long strip $0 \leq x < \infty$, $0 \leq y \leq d$, satisfies the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Given that $T = 0$ at both $y = 0$ and $y = d$, and that $T \longrightarrow 0$ as $x \longrightarrow \infty$, show that the temperature distribution can be written in the form

$$T(x, y) = \sum_{n=1}^{\infty} B_n e^{-n\pi x/d} \sin{(n\pi y/d)}.$$

Hence determine T if the edge at $x = 0$ is held at constant temperature T_0.

Extra: How does the solution change if the edge at $x = 0$ has the following alternative temperature profiles: (i) y; (ii) $y(d - y)$; (iii) $\delta(y - d/2)$ (i.e. the unit impulse based half-way along the interval)?

Q2. The equation governing the time-dependent diffusion of heat in a thin bar of length d is

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}.$$

If the temperature distribution at $t = 0$ is given by $\theta = x^2(d - x)$, and the ends of the bar (which are situated at $x = 0$ and $x = d$) are maintained at zero temperature for all time, determine the subsequent evolution of the temperature field in terms of a Fourier sine series.

Q3. Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where $y = y(x, t)$ is the displacement, subject to the boundary conditions, $y = 0$ on $x = 0$ and $x = 2$, and the initial conditions $y = 2x - x^2$ and $\partial y/\partial t = 0$ at $t = 0$.

Q4. Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

subject to the boundary conditions, $y = 0$ on $x = 0$ and $x = 2$, and the initial conditions $y = 0$ and $\partial y/\partial t = x(2 - x)$ at $t = 0$. Note that this isn't the same question as Q3, although it looks very similar.

Q5. Small-amplitude free transverse vibrations $y(x,t)$ of a simply supported uniform beam of length d satisfy the equation

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

and are subject to the boundary conditions $y = 0$ and $\frac{\partial^2 y}{\partial x^2} = 0$ at both $x = 0$ and $x = d$. Given that the beam has initial displacement $y = x(d-x)$ and zero velocity at $t = 0$, show that the subsequent motion is given by

$$y(x,t) = \frac{8d^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\Big(\frac{(2n+1)\pi x}{d}\Big)\cos\Big(\frac{(2n+1)^2\pi^2 ct}{d^2}\Big).$$

Q6. The equation governing small transverse displacements $y(x,t)$ of a uniform string stretched between two fixed points a distance d apart and subject to light damping is

$$\frac{\partial^2 y}{\partial t^2} + 2k\frac{\partial y}{\partial t} = c^2\frac{\partial^2 y}{\partial x^2}.$$

Given that $y = 0$ at both $x = 0$ and $x = d$ show that the general motion of the string can be written in the form

$$y(x,t) = e^{-kt}\sum_{n=1}^{\infty}\Big[A_n\cos(d_n t) + B_n\sin(d_n t)\Big]\sin\left(\frac{n\pi x}{d}\right)$$

where

$$d_n^2 = \frac{n^2\pi^2c^2}{d^2} - k^2.$$

Assume that the string is sufficiently lightly damped that $(\pi^2c^2/d^2 - k^2)$ is positive. If the motion of the string is initiated by giving it an initial displacement $y = \sin(\pi x/d)$ at $t = 0$ with zero velocity, determine an expression for the subsequent motion of the string.

Note that you will find some worked examples of this type of problem in chapter 13 of "Advanced Engineering Mathematics" by P. V. O'Neil. Other good text books are (i) "Advanced Engineering Mathematics" by Wylie and Barrett (McGraw Hill) and "Applied Fourier Analysis" by H. P. Hsu, in the Harcourt Brace Jovanovich College Outline Series, although the latter is a very advanced book.

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S2 Modelling Techniques   Sheet 3

Separation of Variables for PDEs and Fourier Series II

Fourier Cosine and Quarter-range Sine Series

Q1. The slides corresponding to Videos 2, 3a and 3b contain quite a few Fourier Sine, Fourier Cosineand Quarter-range Fourier Sine series without proof. Use these to practice your integration by parts! Specifically:

pp. 18 and 19 in: `https://people.bath.ac.uk/ensdasr/ME20021.bho/mt2.slides2.pdf`

pp. 5, 6, 7 and 12 in: `https://people.bath.ac.uk/ensdasr/ME20021.bho/mt2.slides3.pdf` .

Q2. The steady temperature distribution $T(x, y)$ inside the infinitely long strip $0 \le x < \infty$, $0 \le y \le d$, satisfies the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,$$

with the following insulating boundary conditions: $\dfrac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = d$, and subject to the boundary conditions,

$$\frac{\partial T}{\partial x} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty \qquad \text{and} \qquad T = y(d - y) \quad \text{on} \quad x = 0.$$

Write the solution in terms of a Fourier cosine series.

Q3. The equation governing the time-dependent diffusion of heat in a thin bar of length d is

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}.$$

If the temperature distribution at $t = 0$ is given by $\theta = d - x$, and the ends of the bar (which are situated at $x = 0$ and $x = d$) are insulated (i.e. $\partial\theta/\partial x = 0$) for all time, determine the subsequent evolution of the temperature field in terms of a Fourier cosine series.

Q4. The equation governing the time-dependent diffusion of heat in a thin bar of length d is

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}.$$

The boundary conditions $\theta = 0$ at $x = 0$ and $\partial\theta/\partial x = 0$ at $x = d$. The solution of this problem uses a quarter–range Fourier series. The initial condition is $\theta = 1$ at $t = 0$.

Q5. If you needed to solve the previous question subject to the boundary conditions that $\partial\theta/\partial x = 0$ at $x = 0$ and $\theta = 0$ at $x = d$, then how would you proceed?

Finite domains with Laplace's equation

Note: that it is essential to sketch the domain the boundary conditions prior to undertaking the Separation of Variables analysis.

Q6. Solve Laplace's equation in the rectangular region, $0 \le x \le 2$ and $0 \le y \le 1$, where T satisfies the boundary conditions,

$$T = 0 \text{ on } y = 0, 1, \quad T = 0 \text{ on } x = 2 \quad \text{and} \quad T = y - y^2 \text{ on } x = 0.$$

Q7. Solve Laplace's equation in the rectangular region, $0 \le x \le 2$ and $0 \le y \le 1$, where T satisfies the boundary conditions,

$$T = 0 \text{ on } x = 0, 2, \quad T = 0 \text{ on } y = 1 \quad \text{and} \quad T = 2x - x^2 \text{ on } y = 0.$$

Q8. Suppose that we wish to swap around the boundary conditions at $x = 0$ and $x = 2$ in Q1. Is there a simple way to write down this new solution given the form of the solution to Q1? In other words, is it possible just to write it down straightaway without any further analysis?

Q9. Use the solutions given in questions 1 and 2 to solve Laplace's equation in the same domain but where the boundary conditions are now,

$$T = 2x - x^2 \text{ on } y = 0, \quad T = y - y^2 \text{ on } x = 0, \quad T = 0 \text{ on both } y = 1 \text{ and } x = 2.$$

Q10. Solve Laplace's equation in the square domain, $0 \le x \le 1$ and $0 \le y \le 1$ where $T = x$ on $y = 0$, $T = 1 - x$ on $y = 1$ with $T = 0$ on both $x = 0$ and $x = 1$. This will involve two summations.

Q11. Solve Laplace's equation in the square domain, $0 \le x \le 1$ and $0 \le y \le 1$ where $T = x$ on $y = 0$, $T = y$ on $x = 0$, $T = 1 + y$ on $x = 1$ and $T = 1 + x$ on $y = 1$. [Hint: while one may find this solution in terms of four separate summations, there is an extremely simple alternative solution. If one sketches the boundary conditions as a height in the third direction then this may give hint!]

Department of Mechanical Engineering, University of Bath

S2 Modelling Techniques   Sheet 4

Separation of Variables for PDEs and Fourier Series III

Polar coordinates and Laplace's equation

In the whole of this sheet (except for the final question) we will be working with the following PDE:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0.$$

Q1. The slides corresponding to Video 4 contain a few more Fourier series solutions, mostly without workings. Again, you may use these to practice your integration by parts should you wish. See

`https://people.bath.ac.uk/ensdasr/ME20021.bho/mt2.slides4.pdf` .

Q2. A lengthy question!.

(a) Suppose that one has a sector of a circle which lies in $0 \le r \le 1$ and $0 \le \theta \le \alpha$. The outer boundary at $r = 1$ has the temperature, $T = f(\theta)$, imposed upon it, and the straight-edge boundaries at $\theta = 0$ and $\theta = \alpha$ are held at $T = 0$. Find the temperature distribution in this sector in terms of the Fourier Sine Series coefficients of $f(\theta)$.

(b) How do these solutions change when the boundary condition at $\theta = \alpha$ changes to $\partial T/\partial\theta = 0$?

(c) In both cases let $f(\theta) = \theta$ and find the solution for general values of α, and for the specific shapes given by $\alpha = \frac{1}{2}\pi$ (quadrant), $\alpha = \pi$ (semicircle) and $\alpha = \frac{3}{2}\pi$ (three quarters of a circle).

Q3. The curved boundary of a semicircle is maintained at the temperature $T = 1$ when $0 \le \theta < \frac{1}{2}\pi$ and at $T = 0$ when $\frac{1}{2}\pi < \theta < \pi$. The straight boundary/boundaries at $\theta = 0$ and $\theta = \pi$ are held at $T = 0$. Find the temperature distribution within the semicircle.

Use this solution to write down the temperature distribution in the region which is external to the semicircle and which satisfies the same boundary conditions.

Q4. In this question we will be considering the solution of Laplace's equation both within and outside of a full circle, and therefore the separation of variables substitution will need sines and cosines that are periodic. The circumference is maintained at $T = f(\theta)$. Solve for the internal temperature distributions in the following cases:

(a) $f(\theta) = \pi^2 - \theta^2$ in the range $-\pi \le \theta \le \pi$.
(b) $f(\theta) = \theta$ in the range $-\pi < \theta < \pi$.
(c) $f(\theta) = \theta$ in the range $0 < \theta < 2\pi$.

For each of these cases, what are the external temperature fields corresponding to the above thermal boundary conditions?

Q5. **Beyond the scope of the course, but the results are *really* interesting!**

The purpose of this question is to show that solutions obtained using the method of separation of variables are not the only solutions which may be obtained. This solution of the wave equation is known as d'Alembert's solution.

The one–dimensional wave equation is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

By substituting into the equation show that $y = f(x - ct) + g(x + ct)$ is a solution where both f and g are arbitrary functions. This does involve some partial differentiation.

(a) Suppose the initial conditions $y = 1/(1 + x^2)$ and $\frac{\partial y}{\partial t} = 0$ are specified at $t = 0$. Find explicit expressions for f and g and hence $y(x, t)$. Sketch the solution at $t = 0$ and at a later time, and give a physical interpretation of the initial conditions and the solution.

(b) Repeat (a) for the initial conditions, $y = 0$ and $\frac{\partial y}{\partial t} = 1/(1 + x^2)$ at $t = 0$.

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S2 Modelling Techniques   Sheet 5

Fourier Transforms: Introductory Parts

Q1. Sketch the following functions and find their Fourier Transforms:

(a) $f(x) = 1$, $(|x| < 0.5)$, $f(x) = 0$, otherwise.
(b) $f(x) = 1$, $(0 < x < 1)$, $f(x) = 0$, otherwise.
(c) $f(x) = 1 - x$, $(0 \le x \le 1)$, $f(x) = 1 + x$, $(-1 \le x \le 0)$, $f(x) = 0$, otherwise.
(d) $f(x) = \sin x$, $(-\pi \le x \le \pi)$, $f(x) = 0$, otherwise.
(e) $f(x) = e^{-x}$, $(-1 < x < 1)$, $f(x) = 0$, otherwise.
(f) $f(x) = -1$, $(-1 < x < 0)$, $f(x) = +1$, $(0 < x < 1)$, $f(x) = 0$, (otherwise).

In all cases, take advantage of any symmetry in the given expression for $f(x)$ to simplify the Fourier Transform integral.

Q2. A unit pulse is defined as being, $P(x) = 1$ when $-\frac{1}{2} < x < \frac{1}{2}$, and it is zero otherwise. Find the convolution, $P(x) * P(x)$. It will be worth sketching both $P(\xi)$ and $P(x - \xi)$ in order to determine where they overlap.

Q3. (A little difficult!) Find the Fourier transform of $f(x)$ where $f(x) = e^{-x}$ for $x > 0$ and $f(x) = 0$ for $x < 0$; this is not the same as Q1e, above.

Use the result that the Fourier transform of the convolution of two functions is the product of their respective transforms (the convolution theorem) to show that the inverse transform of $1/(1 + \omega j)^2$ is a function which is equal to xe^{-x} for $x > 0$ and equal to zero for $x < 0$. (Hint: you will need to be particularly careful about where the functions in the convolution integral are nonzero, and to take this into account when modifying the limits of integration.)

Q4. Use some suitable Fourier Transforms from Q1 together with the Symmetry Theorem to find the Fourier Transforms of the following functions:

$$\text{(i)}\ \frac{\sin x/2}{x/2}; \qquad \text{(ii)}\ \frac{\sin \pi x}{x^2 - 1}; \qquad \text{(iii)}\ \frac{1 - \cos x}{x}; \qquad \text{(iv)}\ \frac{\sin x}{x}.$$

In the last case you will need to find the function of x whose FT is $(\sin \omega)/\omega$.

Q5. Suppose $f(x)$ represents a transmitted signal. This signal is *frequency modulated* when it is multiplied by a sinusoidal signal of frequency ω_c. Use the definition of the Fourier transform to prove the Frequency Modulation theorem:

$$\mathcal{F}[f(x) \cos \omega_c x] = \Big[F(\omega + \omega_c) + F(\omega - \omega_c)\Big]/2.$$

Q6. [Note that this is a t-dependent example, not an x-dependent one, but it may be solved in exactly the same way.]

The function $f(t)$ is equal to $e^{-at}\sin bt$ when $t > 0$, and is zero otherwise. Show that $\mathcal{F}[\{(\sqcup)]$ is given by

$$\mathcal{F}[f(t)] = \frac{b}{(\omega j)^2 + 2a(\omega j) + (a^2 + b^2)}.$$

Use all the necessary results given in the lecture notes to show that the following forced mass spring damper system,

$$y'' + 2ay' + (a^2 + b^2)y = g(t),$$

where both a and b are positive, has a solution which can be written in the form

$$y(t) = \frac{1}{b}\int_0^\infty g(t-\tau)e^{-a\tau}\sin b\tau \, d\tau.$$

If the forcing function were $g(t) = \delta(t)$, the unit impulse at $t = 0$, then what is the solution?

Q7. Write down the definition of the Fourier transform of $f(x)$ and differentiate it twice with respect to ω to obtain

$$\mathcal{F}[x^2 f(x)] = -\frac{d^2F}{d\omega^2}.$$

Use this result and the time differentiation result given in the lecture notes to obtain the Fourier transform of the equation

$$\frac{d^2 f}{dx^2} - (1 + x^2)f = 0.$$

Would you use Fourier transforms to solve this problem?

Q8. This question will involve solving another time-dependent equation, and so the notation (t) will be slightly different from in the lecture notes (x). You will also need to assemble some armoury before solving the ODE at the end. Two of these are the Fourier Transforms of both $\delta(t-a)$ and $H(t)e^{-t}$. The other two are the formulae (i) for the FT of a single derivative and (ii) for what was called the x-shift theorem in the lectures, but which will now called the t-shift theorem for this question. Finally, we need to define the following function,

$$Ш(t) = \sum_{n=-\infty}^{\infty} \delta(t-n).$$

This is the Shah function which is named after the Cyrillic letter, sha, which it resembles. It is also called the Dirac comb, the bed-of-nails function or, somewhat unimaginatively, as a train of unit impulses. No doubt you can now imagine it!

The objective is solve the ODE, $\boxed{y' + y = Ш(t)}$ using Fourier Transforms. Begin by finding the FT of the Shah function using its definition, and then find the FT of the ODE, eventually solving for $Y(\omega)$, which is the FT of $y(t)$. So $Y(\omega)$ should be in the form of an infinite sum, the inverse FT of which may be found using the t-shift theorem.

What does the solution, $y(t)$, look like?

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Modelling Techniques 2 ME20021 Sheet 6

Fourier Transforms, Fourier Sine and Cosine Transforms

Note: Questions 1 to 3 may be undertaken after Lecture/Video 7, while the remaining may be tackled after Lecture/Video 8.

1. The displacement of an infinitely long string ($-\infty < x < \infty$) satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where $y = f(x)$ and $\partial y/\partial t = 0$ are the given displacement and velocity at $t = 0$. Using Fourier Transforms, show that $Y(\omega, t)$, the Fourier Transform of $y(x, t)$ with respect to x, is given by

$$Y(\omega, t) = \frac{F(\omega)}{2}\Big[e^{j\omega ct} + e^{-j\omega ct}\Big],$$

where $F(\omega) = \mathcal{F}[f(x)]$. Use the Inverse Fourier Transform formula and one of the shift theorems to show that

$$y(x, t) = \Big[f(x - ct) + f(x + ct)\Big]/2.$$

What is the physical meaning of this solution?

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2. Solve Fourier's equation,

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},$$

subject to the initial condition that $\theta = e^{-|x|}$ at $t = 0$, and the boundary conditions, $\theta \to 0$ as $|x| \to \infty$.

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3. We'll be solving the wave equation again in this question, where the equation itself is given above in Q1. First, though, find the Fourier Transform of the pulse, $P(x; a)$, which is defined as being $P(x; a) = 1$ when $-a < x < a$ and $P(x; a) = 0$ otherwise. (Don't worry about the semicolon; this merely says that a is a parameter.) Now use this result in a solution of the wave equation where $y = 0$ and $\partial y/\partial t = \delta(x)$ at $t = 0$. Attempt a 3D sketch of the resulting solution.

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4. Sketch the following functions and find both their Fourier Sine and Cosine Transforms:
   (a) $f(x) = 1$, $(0 \le x < 1)$, $f(x) = 0$, otherwise.
   (b) $f(x) = 1 - x$, $(0 \le x \le 1)$, $f(x) = 0$, otherwise.
   (c) $f(x) = e^{-x}$, (d) $\delta(x - 1)$.

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5. The symmetry theorem for Fourier Transforms states that, if $F(\omega) = \mathcal{F}[f(x)]$, then the Fourier Transform of $F(x)$ is given by $\mathcal{F}[F(x)] = 2\pi f(-\omega)$. What are the analogous formulae for the

Fourier sine and cosine Transforms? Use these formulae and the appropriate answers to question 4 to find the following, $\mathcal{F}_c[1/(1+x^2)]$, $\mathcal{F}_s[x/(1+x^2)]$, $\mathcal{F}_c[\cos x]$ and $\mathcal{F}_s[\sin x]$.

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6. A lagged, semi-infinite rod is initially at a zero temperature throughout its length $0 \leq x < \infty$. When $t > 0$, its end at $x = 0$ is maintained at a constant temperature θ_0. The temperature $\theta(x,t)$ satisfies Fourier's equation as given in Q2.

Use the Fourier Sine Transform with respect to x to show that the evolution of the temperature field is given by

$$\theta(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\theta_0}{\omega}(1 - e^{-\omega^2 \alpha t}) \sin \omega x \, d\omega.$$

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7. Repeat Question 6 for the case where the initial condition is $\theta(x,0) = e^{-x}$ and where the $x = 0$ end of the rod is insulated, that is, when

$$\frac{\partial \theta}{\partial x}(0,t) = 0.$$

You will need to use the Fourier Cosine Transform.

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8. Steady two-dimensional conduction satisfies Laplace's equation:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

A conducting solid occupies the quarter plane given by $0 \leq x < \infty$ and $0 \leq y < \infty$. The $y = 0$ boundary is maintained at the temperature, $\theta = 0$, while the temperature of the $x = 0$ boundary is $\theta = 1$. Use the Fourier Sine Transform with respect to x to show that the temperature field is given by,

$$\theta = \frac{2}{\pi}\int_0^\infty \frac{(1 - e^{-\omega y})}{\omega} \sin \omega x \, d\omega.$$

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9. The temperature of a solid quarter-plane satisfies Laplace's equation, as given in Question 8. The $y = 0$ boundary is held at the temperature $\theta = e^{-x}$ while the $x = 0$ boundary is insulated (i.e. the x-derivative of θ is zero). Use the Fourier Cosine Transform to show that the temperature is given by,

$$\theta = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\omega^2} e^{-\omega y} \cos \omega x \, d\omega.$$

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10. The displacement, $y(x,t)$, of a taut elastic string satisfies the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

The string lies in the region $0 \le x < \infty$. Suppose the string is in equilibrium for $t < 0$, i.e. that it has zero displacement and velocity. Suppose also that, at $t = 0$, the $x = 0$ end of the string is raised instantaneously to the new value, $y = 1$, and is held there for all time. Use the Fourier Sine Transform to show that the subsequent displacement of the string is given by,

$$y = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos c\omega t)}{\omega} \sin \omega x \, d\omega.$$

The analytical solution for y may also be written down in a very simple form: $y = H(ct - x)$, where H is the unit step function. Find its Fourier Sine Transform with respect to x to confirm that your solution was correct prior to applying the inverse Fourier Sine Transform.