ME20021 MODELLING TECHNIQUES 2 — Video 7

This video covers:

- A Fourier Transform solution of Fourier's equation.
- A Fourier Transform solution of Laplace's equation

When necessary, use will be made of results from the first two videos.

7 Fourier Transform solutions of PDEs

7.1 Solutions of Fourier's equation

We will solve Fourier's equation,

$$\boxed{\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2},} \qquad \text{subject to} \qquad \boxed{\theta = f(x) \quad \text{at} \quad t = 0.}$$

We assume that $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ for the FT to exist.

The general procedure for solving PDEs follows the three steps:

(i) take the Fourier Transform of the given equation,

(ii) solve the transformed equation,

(iii) take the inverse Fourier Transform to find the desired solution.

Note: In many cases the final answer has to be written in terms of an integral, but sometimes one may be able to find an explicit expression for the solution.

Define $\Theta(\omega, t)$ to be the FT of $\theta(x, t)$ with respect to x:

$$\boxed{\Theta(\omega, t) = \mathcal{F}\left[\theta(x, t)\right] = \int_{-\infty}^{\infty} \theta\, e^{-j\omega x}\, dx.}$$

Now we'll take the Fourier Transform of the second derivative:

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial^2 \theta}{\partial x^2}\right] &= \int_{-\infty}^{\infty} \frac{\partial^2 \theta}{\partial x^2} e^{-j\omega x}\, dx \\
&= \cancel{\Big[\underbrace{\frac{\partial \theta}{\partial x}}_{I_1}\Big]}\Big[\underbrace{e^{-j\omega x}}_{D_0}\Big]_{-\infty}^{\infty} - \cancel{\Big[\underbrace{\theta}_{I_2}\Big]}\Big[\underbrace{-j\omega e^{-j\omega x}}_{D_1}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\Big[\underbrace{\theta}_{I_2}\Big]\Big[\underbrace{-\omega^2 e^{-j\omega x}}_{D_2}\Big]\, dx \\
&= -\omega^2 \Theta.
\end{aligned}$$

Note: that we have used the facts that both θ and its x-derivative tend to zero as $x \to \pm\infty$.

The Fourier Transform of the time-derivative term is relatively straightforward:

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial \theta}{\partial t}\right] &= \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t}\, e^{-j\omega x}\, dx \\
&= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \theta\, e^{-j\omega x}\, dx \qquad \text{swapped } \frac{\partial}{\partial t} \text{ and } \int dx \\
&= \frac{\partial \Theta}{\partial t}.
\end{aligned}$$

Note that we have assumed that the integral of the derivative is the same as the derivative of the integral.

Now Fourier's equation is transformed:

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \xrightarrow{FT} \boxed{\frac{\partial \Theta}{\partial t} = -\alpha \omega^2 \Theta.}$$

This ordinary differential equation has the solution,

$$\Theta = A(\omega) e^{-\alpha \omega^2 t}.$$

Weird: The arbitrary constant, $A(\omega)$, is allowed to be a function of ω because we have solved an ODE in t.

$$\Theta = A(\omega)e^{-\alpha\omega^2 t}. \tag{1}$$

Now we need to apply the Initial Condition.

We had $\boxed{\theta = f(x) \text{ at } t = 0,}$ and therefore the Fourier Transform of this is,

$$\Theta = \mathcal{F}\left[f(x)\right] = F(\omega) \quad \text{at} \quad t = 0. \tag{2}$$

Comparison of (1) at $t = 0$ and (2) yield $\boxed{A(\omega) = F(\omega).}$

Hence, the Fourier Transform of the desired solution is,

$$\boxed{\Theta = F(\omega)e^{-\alpha\omega^2 t}.}$$

Now apply the formula for the inverse Fourier Transform,

$$\boxed{\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha\omega^2 t}e^{j\omega x}\, d\omega.}$$

$$\boxed{\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha\omega^2 t}e^{j\omega x}\,d\omega.} \tag{3}$$

Some simplifications.

One simplification is when $f(x)$ is an even function. Check this out:

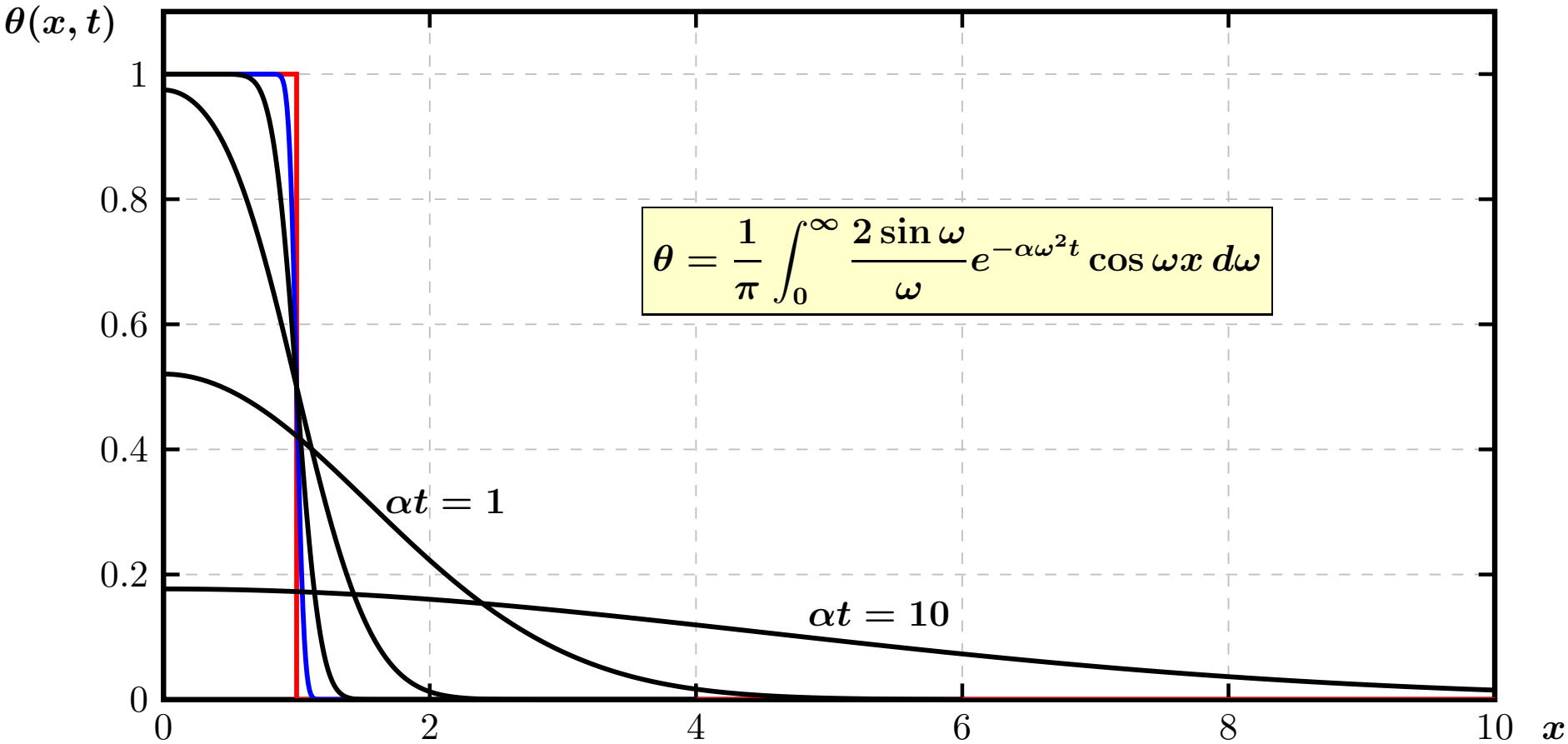
$$\begin{aligned}
F(\omega) = \mathcal{F}\left[f(x)\right] &= \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx \\
&= \int_{-\infty}^{\infty} f(x)\Big(\cos\omega x - j\cancel{\sin\omega x}\Big)\,dx && f(x)\sin\omega x \text{ is odd} \\
&= \int_{-\infty}^{\infty} f(x)\cos\omega x\,dx. && \boxed{\text{So } F(\omega) \text{ is even.}}
\end{aligned}$$

In the same way, then, Eq. (3) above may be simplified:

$$\begin{aligned}
\theta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha\omega^2 t}e^{j\omega x}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha\omega^2 t}\cos\omega x\,d\omega && \text{removing the odd component} \\
&= \boxed{\frac{1}{\pi}\int_{0}^{\infty} F(\omega)e^{-\alpha\omega^2 t}\cos\omega x\,d\omega} && \text{integrand is even in } \omega
\end{aligned}$$

Now we'll use the following initial condition:

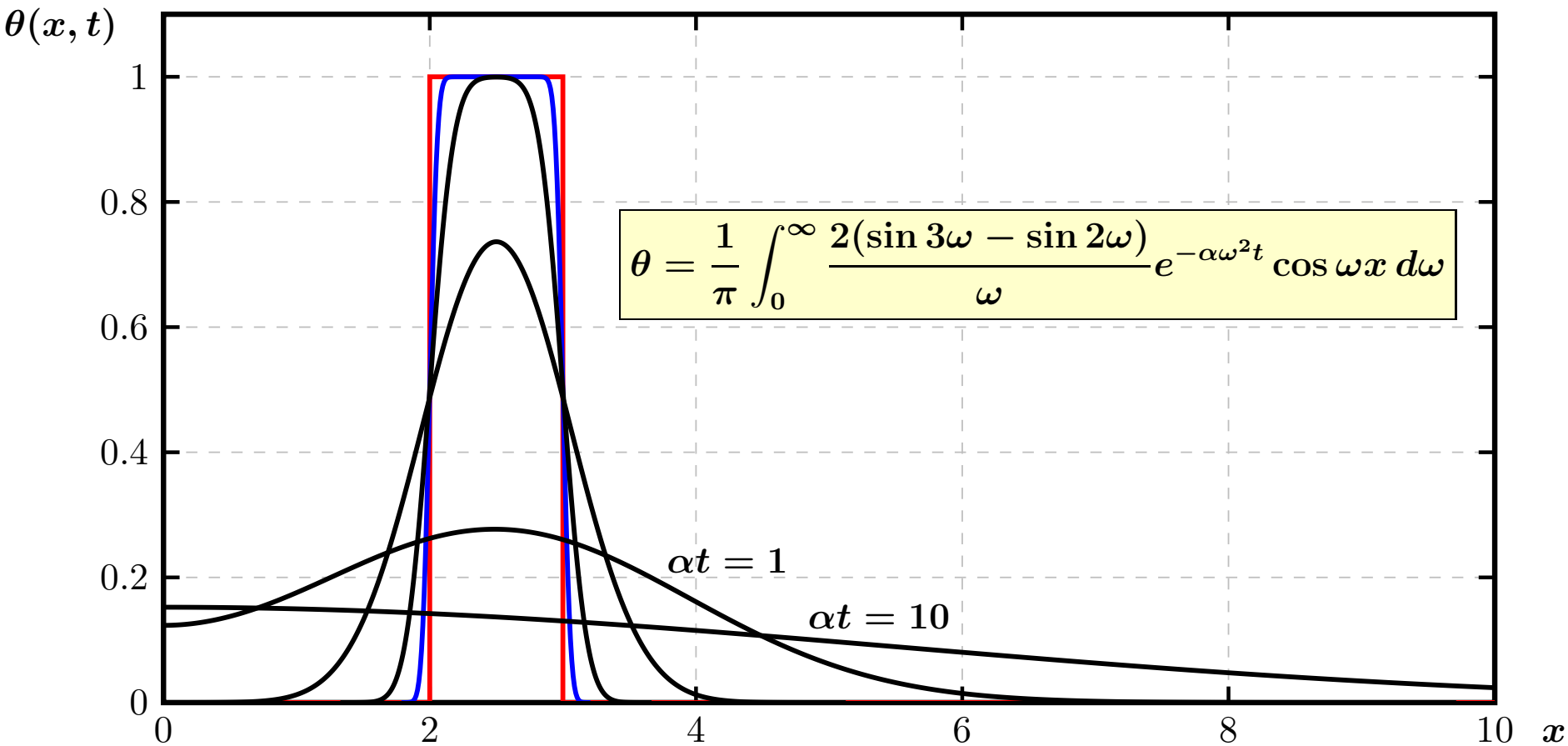
$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases} \qquad \Longrightarrow \qquad F(\omega) = \frac{2\sin\omega}{\omega},$$



Successive temperature profiles for $\alpha t = 0$ (red), $\alpha t = 0.001$, (blue), $\alpha t = 0.01$, $\alpha t = 0.1$, $\alpha t = 1$ and $\alpha t = 10$. Displaying the region $x \geq 0$ only; the solutions are even in x.

Two pulses

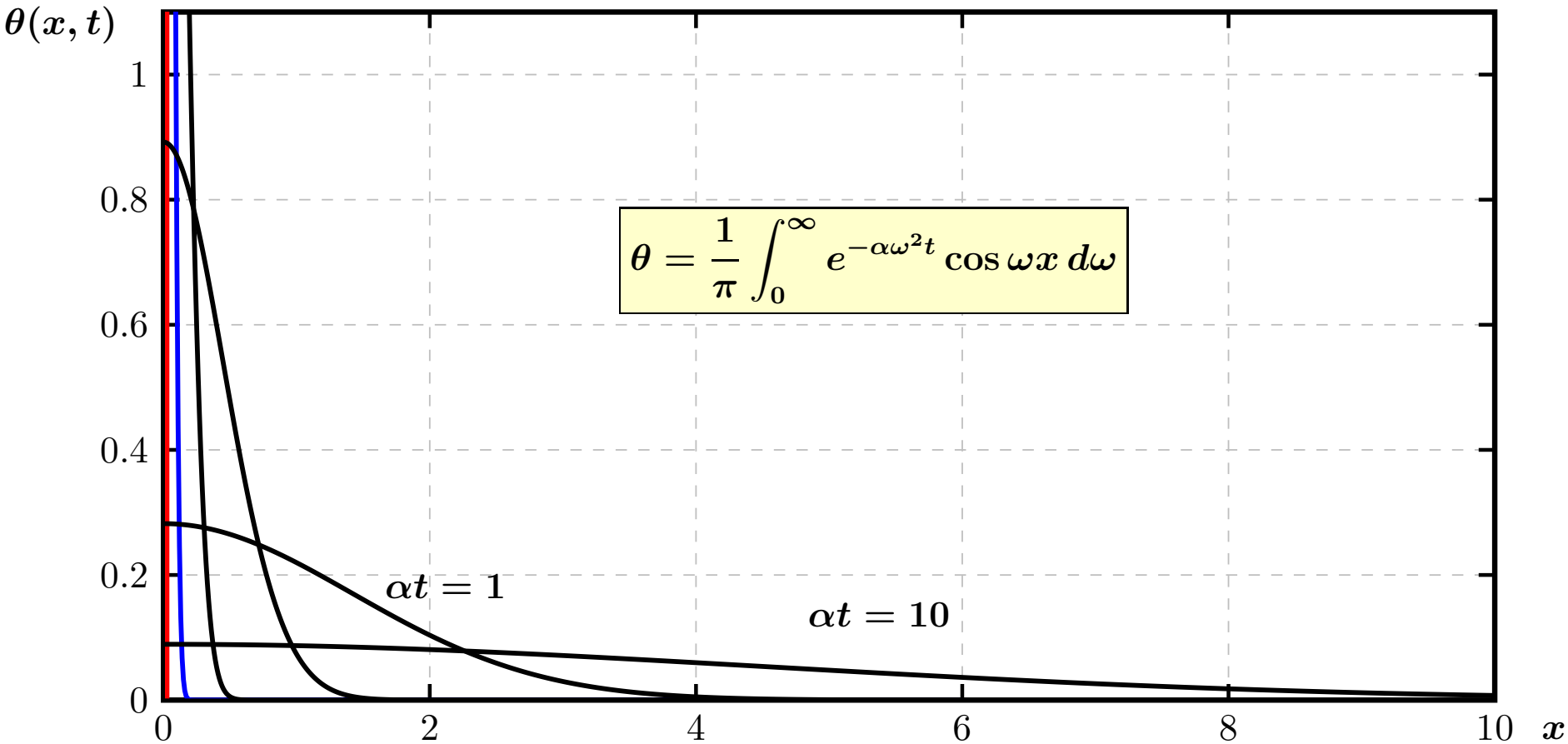
$$f(x) = \begin{cases} 1 & \text{for } \ 2 < |x| < 3 \\ 0 & \text{otherwise} \end{cases} \qquad \Longrightarrow \qquad F(\omega) = \frac{2(\sin 3\omega - \sin 2\omega)}{\omega},$$



Successive temperature profiles for $\alpha t = 0$ (red), $\alpha t = 0.001$, (blue), $\alpha t = 0.01$, $\alpha t = 0.1$, $\alpha t = 1$ and $\alpha t = 10$. Displaying the region $x \geq 0$ only; the solutions are even in x.

Another simplification is having a unit impulse. This one is at the origin.

$$f(x) = \delta(x) \qquad \Longrightarrow \qquad F(\omega) = 1.$$



Successive temperature profiles for $\alpha t = 0$ (red), $\alpha t = 0.001$, (blue), $\alpha t = 0.01$, $\alpha t = 0.1$, $\alpha t = 1$ and $\alpha t = 10$. Displaying the region $x \geq 0$ only; the solutions are even in x.

Note: Also see the problem sheet for a further simplification of this solution.

A curiosity: a conservation principle

In these problems there is no place for the heat to escape to, and it seems reasonable that the total heat content must be conserved. Let's check it out mathematically:

The total heat content (relative to the ambient) is $\boxed{\int_{-\infty}^{\infty} \theta \, dx.}$

If we integrate Fourier's equation in the same way then,

$$\int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} \, dx = \int_{-\infty}^{\infty} \frac{\partial^2 \theta}{\partial x^2} \, dx$$

$$\Longrightarrow \quad \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} \theta \, dx \right] = \left[\frac{\partial \theta}{\partial x} \right]_{-\infty}^{\infty}$$

$$\Longrightarrow \quad \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} \theta \, dx \right] = 0.$$

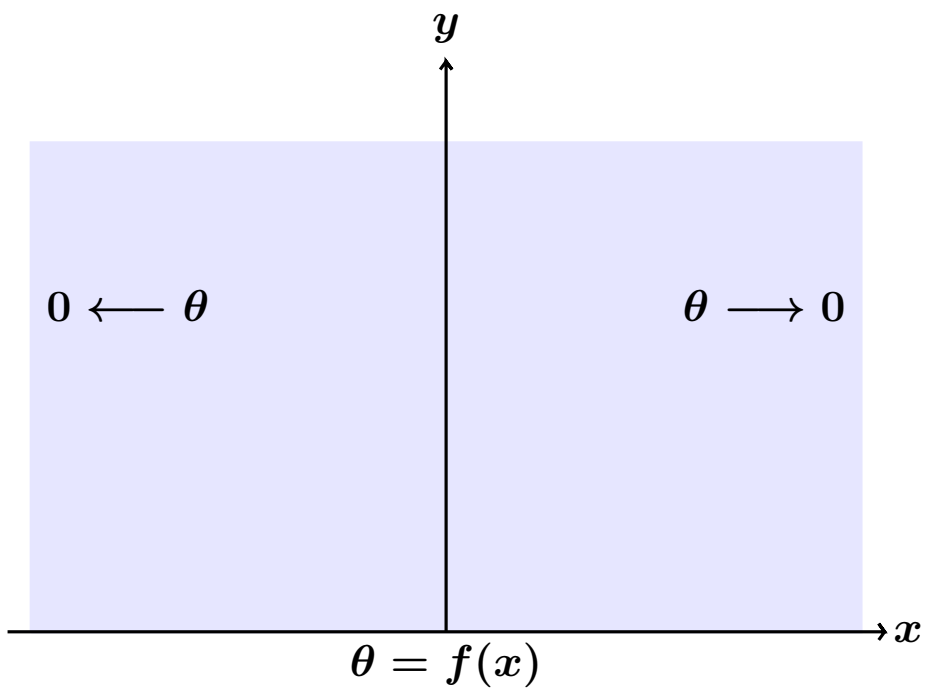
So no heat is lost because there are no boundaries through which it can be lost.

7.2 Solution of Laplace's equation

We now solve

$$\boxed{\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0,}$$

in the ranges $-\infty < x < \infty$ and $0 \leq y < \infty$ — steady 2D conduction in a semi-infinite plane.



It is assumed that $\boxed{\theta \to 0 \text{ and } f(x) \to 0 \text{ as } x \to \pm\infty}$ so that FTs may be used.

Now we take Fourier Transforms with respect to x:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \qquad \xrightarrow{\quad FT \quad} \qquad \frac{\partial^2\Theta}{\partial y^2} - \omega^2\Theta = 0.$$

The general solution of this ODE is,

$$\boxed{\Theta = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y}.}$$

Now $\theta \to 0$ as $y \to 0$, which implies that $\Theta \to 0$ too.

Note: In the inverse FT formula, ω is integrated between $-\infty$ and $+\infty$.

When $\omega > 0$ we need $A = 0$ so that $\Theta = B(\omega)e^{-\omega y}$
When $\omega < 0$ we need $B = 0$ so that $\Theta = A(\omega)e^{\omega y}$.

This avoids exponential growth.

$$\boxed{\begin{array}{l}\text{When } \omega > 0 \text{ we need } A = 0 \text{ so that } \Theta = B(\omega)e^{-\omega y} \\ \text{When } \omega < 0 \text{ we need } B = 0 \text{ so that } \Theta = A(\omega)e^{\omega y}.\end{array}}$$

We may combine these two cases into one formula in the following way:

$$\boxed{\Theta = C(\omega)e^{-|\omega|y} \qquad \text{where } C(\omega) \text{ is arbitrary.}} \quad \text{Weird again....} \tag{1}$$

Boundary condition:

$$\theta = f(x) \text{ at } y = 0 \qquad \Longrightarrow \qquad \boxed{\Theta = F(\omega) \quad \text{when } y = 0.} \tag{2}$$

Equations (1) and (2) imply that $C = F$, and hence the transformed solution is

$$\boxed{\Theta(\omega, y) = F(\omega)\, e^{-|\omega|y}.}$$

Now use the inverse Fourier Transform.

$$\boxed{\theta(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{-|\omega|y}\, e^{j\omega x}\, d\omega.}$$

Or....

$$\boxed{\Theta(\omega, y) = F(\omega)\, e^{-|\omega| y}.}$$

This is the product of two functions of ω, and therefore the Convolution Theorem may be used.

We know that $\mathcal{F}^{-1}[F(\omega)] = f(x)$. But what about $\mathcal{F}^{-1}[e^{-|\omega| y}]$?

Earlier we used the Symmetry Theorem to show that

$$\mathcal{F}[\underbrace{e^{-|x|}}_{f(x)}] = \underbrace{\frac{2}{1+\omega^2}}_{F(\omega)} \qquad \Longrightarrow \qquad \mathcal{F}\Big[\underbrace{\frac{2}{1+x^2}}_{F(x)}\Big] = \underbrace{2\pi e^{-|\omega|}}_{2\pi f(-\omega)} ,$$

but this isn't quite what we need.

So we'll try with $\mathcal{F}[e^{-a|x|}]$ instead.

Avoiding some of the details of the integration (sorry!) we get,

$$\begin{aligned}
\mathcal{F}[e^{-a|x|}] &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-j\omega x}\, dx \\
&= \int_{-\infty}^{\infty} e^{-a|x|} \cos \omega x\, dx && \text{imaginary component is odd} \\
&= 2\int_{0}^{\infty} e^{-a|x|} \cos \omega x\, dx && \text{integrand is even} \\
&= \frac{2a}{a^2+\omega^2}. && \text{Using I.by.P. or the complex trick}
\end{aligned}$$

The Symmetry Theorem is: $\boxed{\mathcal{F}\left[f(x)\right] = F(\omega) \qquad \Longrightarrow \qquad \mathcal{F}\left[F(x)\right] = 2\pi f(-\omega).}$

Hence, $$\mathcal{F}[e^{-a|x|}] = \frac{2a}{a^2+\omega^2} \qquad \Longrightarrow \qquad \mathcal{F}\left[\frac{2a}{a^2+x^2}\right] = 2\pi e^{-a|\omega|}.$$

We may divide by 2π and let $a = y$:

$$\boxed{\mathcal{F}\left[\frac{y/\pi}{y^2+x^2}\right] = e^{-|\omega|y}} \qquad \Longrightarrow \qquad \boxed{\mathcal{F}^{-1}\left[e^{-|\omega|y}\right] = \frac{y/\pi}{y^2+x^2}.}$$

Recall:

$$\boxed{\mathcal{F}^{-1}\left[e^{-|\omega|y}\right] = \frac{y/\pi}{y^2+x^2}.} \qquad \boxed{\Theta(\omega, y) = F(\omega)\, e^{-|\omega|y}.}$$

$$\boxed{f * g = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\, d\xi = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi} \qquad \boxed{\mathcal{F}\left[f * g\right] = F(\omega)G(\omega)}$$

So if $G(\omega, y) = e^{-|\omega|y}$ then $g(x, y) = \dfrac{y/\pi}{y^2+x^2}$.

Using all five of the above results we find that,

$$\begin{aligned} \theta &= f(x) * \left[\frac{y/\pi}{y^2+x^2}\right] \\ &= \boxed{\int_{-\infty}^{\infty} f(x-\xi)\left[\frac{y/\pi}{y^2+\xi^2}\right] d\xi.} \end{aligned}$$

Clearly this provides an alternative integral expression for the final solution to the one we found earlier.

$$\boxed{\theta = \int_{-\infty}^{\infty} f(x-\xi)\left[\frac{y/\pi}{y^2+\xi^2}\right] d\xi.}$$

Generally, that is as far as we can go.....unless $f(x) = \delta(x)$. Therefore,

$$\boxed{\theta = \frac{y/\pi}{y^2+x^2}.}$$

Before looking at a contour plot, let us check this formula.

- When $y = 0$ we have $\theta = 0$ which is consistent with $\theta = \delta(x)$, except perhaps at the origin.
- When $x = 0$ we have $\theta = 1/(\pi y)$, which is infinite at the origin.
- If we move radially outward from the origin (use $x = r\cos\phi$ and $y = r\sin\phi$) we have

  $$\theta = \frac{\sin\theta}{\pi r}$$

  and so the temperature field decays with distance from the origin.

$$\boxed{\theta = \frac{y/\pi}{y^2 + x^2}.}$$

Astonishingly, contours of $\theta =$ constant are circles! Let's check this out:

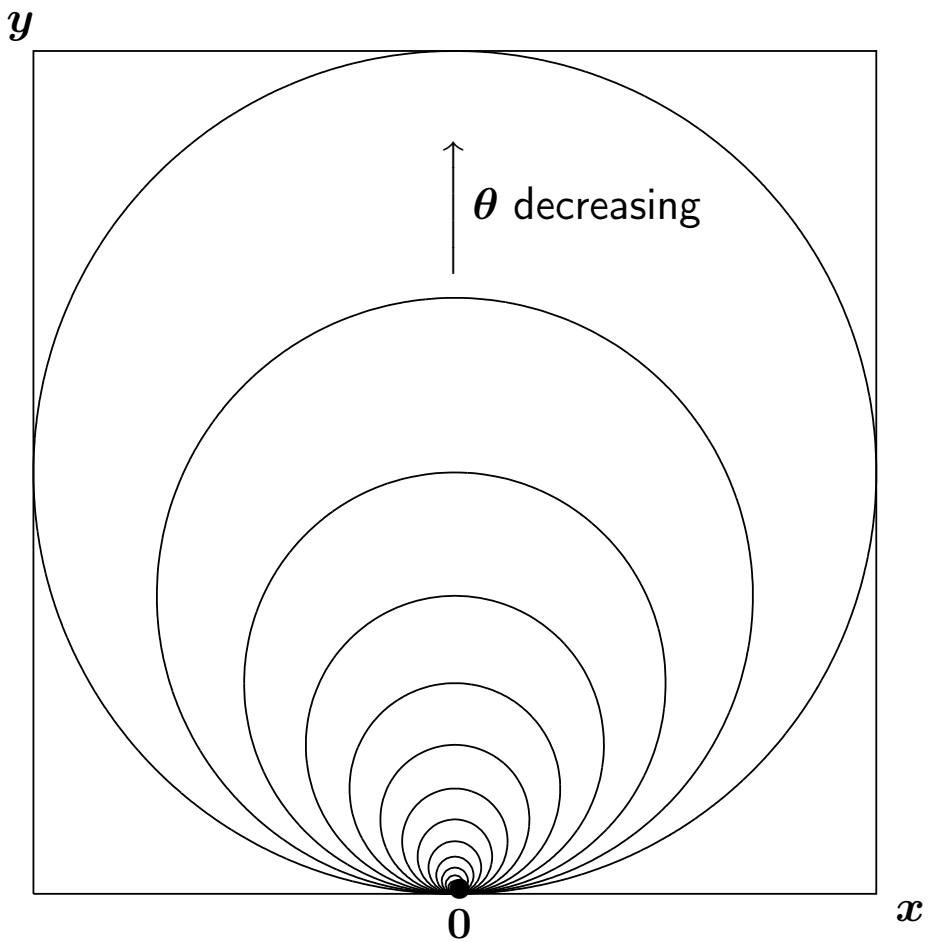
- If we let $\theta = 1/(2\pi\alpha)$, then,

$$\frac{y}{\pi(x^2 + y^2)} = \frac{1}{2\pi\alpha} \qquad \Longrightarrow \qquad x^2 + y^2 = 2\alpha y \qquad \Longrightarrow \qquad x^2 + (y - \alpha)^2 = \alpha^2.$$

- This is a circle of radius α which is centred at $(x, y) = (0, \alpha)$.

- All the circles pass through the origin!

[A secret: when I did this analysis first, I used $\theta = c$, and it eventually dawned on me that $c = 1/(2\pi\alpha)$ is much better. But it is an *a posteriori* thing, and hindsight is marvellous.]



Depicting the contours (i.e. isotherms) of the temperature field due to a unit impulse at the origin.

In summary.

- The process of finding a general solution for these two PDEs doesn't take long.
- It's the additional stuff concerning the Boundary Condition that is more technical, but only when it is possible to simplify the solution.
- It is in these technicalities that most of the results of the first two FT videos are used.
- There will be a question on the wave equation in the associated problem sheet, with a comprehensive solution.

Coming next: the Fourier Sine Transform and the Fourier Cosine Tranform — don't fear, there isn't much difference.