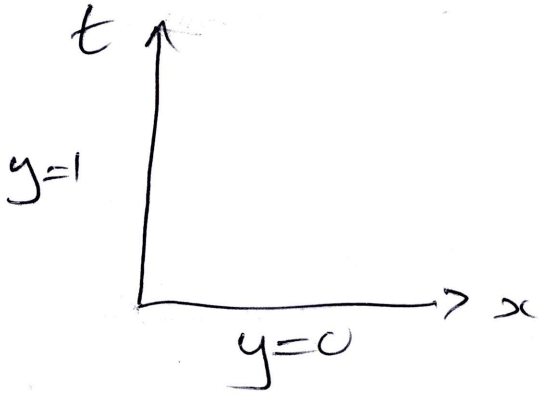
21/22 Q4

$$\mathcal{L}_s[f''] = \omega f(0) - \omega^2 F_s(\omega)$$

(b) $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ | $t=0 \Rightarrow y=0$, $x=0 \Rightarrow y=1$

$\mathcal{L}_s[\quad]$



$$\mathcal{L}_s\left[\frac{\partial^2 y}{\partial t^2}\right] = \int_0^\infty \frac{\partial y}{\partial t} \sin \omega x \, dx$$

$$= \frac{\partial^2}{\partial t^2}\int_0^\infty y \sin \omega x \, dx = \frac{\partial^2 Y_s}{\partial t^2}$$

$$\mathcal{L}_s\left[\frac{\partial^2 y}{\partial x^2}\right] = \omega y(0) - \omega^2 Y_s$$

$$= \omega - \omega^2 Y_s.$$

$$\Rightarrow \frac{\partial^2 Y_s}{\partial t^2} = c^2\left[\omega - \omega^2 Y_s\right]$$

$$\Rightarrow \boxed{\frac{\partial^2 Y_s}{\partial t^2} + \omega^2 c^2 Y_s = c^2\omega.}$$

$\lambda = \pm \omega c j$

$$\Rightarrow Y_s = \underbrace{A(\omega)\cos \omega c t + B(\omega)\sin \omega c t}_{C.F.} + \underbrace{\frac{1}{\omega}}_{P.I.} \quad \text{— (1)}$$

I.C. $y=0$ at $t=0$ $\Rightarrow Y_s = 0$

$\frac{\partial y}{\partial t}=0$ at $t=0$, & $\frac{\partial Y_s}{\partial t}=0$.

① $\overset{t=0}{\Rightarrow}$ $0 = A + \frac{1}{\omega}$ $\Rightarrow A = -1/\omega$.

Also $\frac{\partial Y_s}{\partial t} = -A(\omega)\,\omega c \sin\omega ct + \omega c B \cos\omega ct$

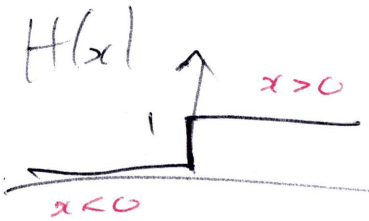
IC $\Rightarrow$ $0 = \omega c B$ $\Rightarrow B = 0$.

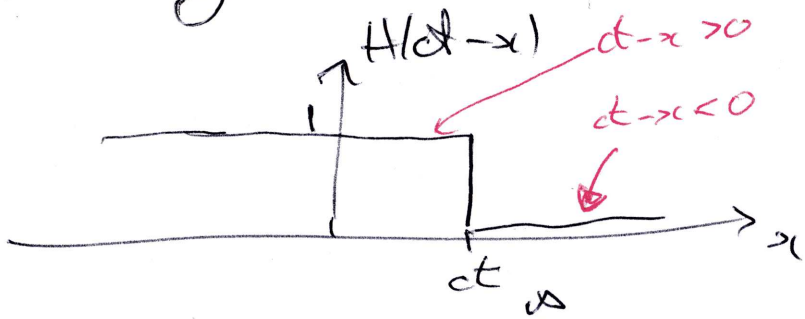
Hence ① $\Rightarrow$ $Y_s = -1/\omega \cos\omega ct + 1/\omega$

$= \frac{1}{\omega}(1-\cos\omega ct)$

I.F.S.T. $\Longrightarrow$ $\boxed{y = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}(1-\cos\omega ct)\sin\omega x\, d\omega.}$

(c) $y = H(ct-x)$.



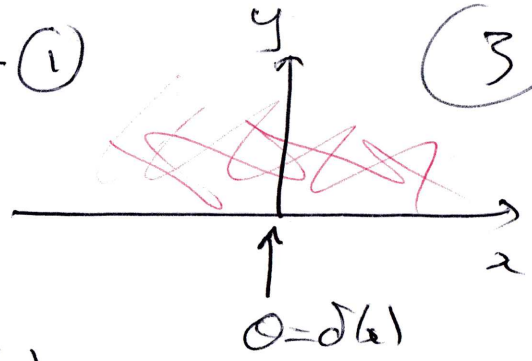


$\mathcal{F}_s[H(ct-x)] = \int_0^\infty H(ct-x)\sin\omega x\, dx$

$= \int_0^{ct} \underbrace{H(ct-x)}_{1}\sin\omega x\, dx + \cancel{\int_{ct}^\infty \underbrace{H(ct-x)}_{0}\sin\omega x\, dx}$

$= -\frac{\cos\omega x}{\omega}\Big|_0^{ct} = \frac{1-\cos\omega ct}{\omega}$.

$$\Theta = F(\omega)\, e^{-|\omega| y} \quad \text{—— (1)}$$



17/18 Q4 (d)

At $y=0$ $\quad \Theta = \delta(x)$.

$$\xrightarrow{y} \Theta = 1$$

$$\int_{-\infty}^{\infty} \underbrace{\delta(x-c)}_{@x=c}\, g(x)\, dx = g(c)$$

$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} \underbrace{\delta(x)}_{@x=0} e^{-j\omega x}\, dx = 1$$

$$(1) \xrightarrow{y=0} 1 = F(\omega)\, e^{-|\omega| 0}$$

$$= F(\omega)$$

$$\Rightarrow F(\omega) = 1.$$

Hence $\boxed{\Theta = e^{-|\omega| y}}$

Symmetry theorem. $\boxed{\mathcal{F}[f(x)] = F(\omega) \Longrightarrow \mathcal{F}[F(x)] = 2\pi f(-\omega)}$

$$\underbrace{f(x)}_{e^{-a|x|}} \qquad \underbrace{F(\omega)}_{\frac{2a}{a^2+\omega^2}} \qquad \underbrace{F(x)}_{\frac{2a}{a^2+x^2}} \qquad \underbrace{2\pi f(-\omega)}_{2\pi e^{-a|-\omega|}}$$

$$\Rightarrow \mathcal{F}^{-1}\left[2\pi e^{-a|\omega|}\right] = \frac{2a}{a^2+x^2}$$

$\div 2\pi$

$a \to y$

$$\Rightarrow \mathcal{F}^{-1}\left[e^{-|\omega| y}\right] = \boxed{\frac{1}{2\pi}\,\frac{2y}{y^2+x^2}}.$$

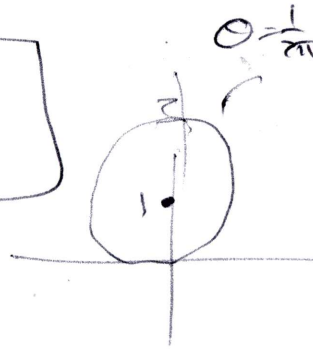
$$\Rightarrow \Theta = \frac{1}{\pi}\,\frac{y}{y^2+x^2}$$

$\Theta = \frac{1}{2\pi}$

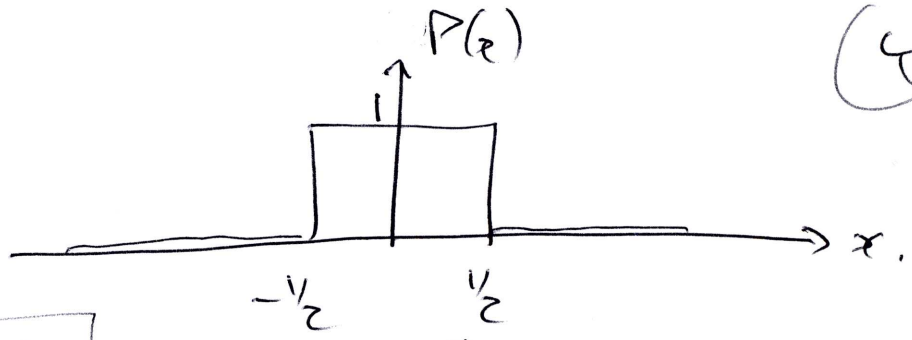
$$\frac{1}{2\pi} = \frac{1}{\pi}\,\frac{y}{y^2+x^2} \Rightarrow (y^2+x^2) = 2y.$$

$$\Rightarrow y^2 - 2y + 1 + x^2 = 0 + 1 \Rightarrow (y-1)^2 + x^2 = 1$$

(+1 each side to get $(y-1)^2$)

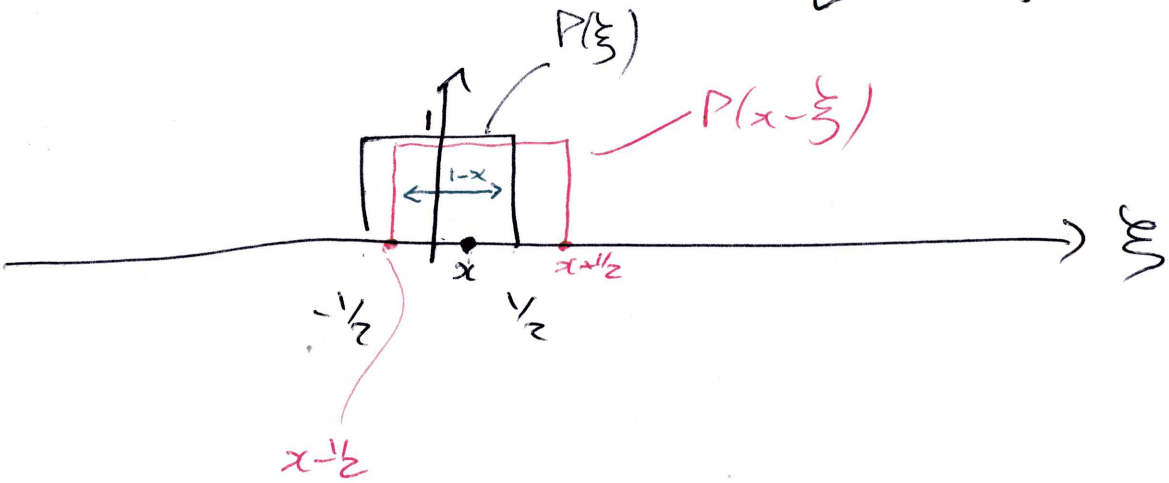


Sheet 5 Q2.



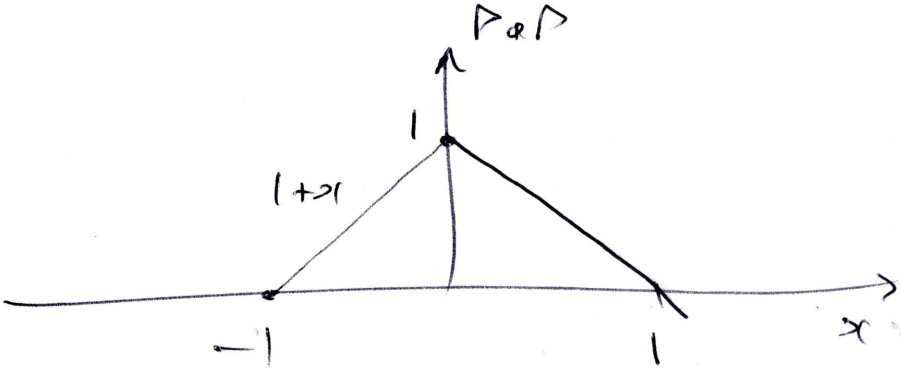
$$\boxed{P(x) * P(x)} = \int_{-\infty}^{\infty} P(\xi)\, P(x-\xi)\, d\xi$$

by [definition]



Pulses overlap by $1-x$ (when $x>0$)

$$\Rightarrow P(x) * P(x) = 1-x \quad (x>0)$$



Hence
$$P * P = \begin{cases} 1-x & (0 \le x \le 1) \\ 1+x & (-1 \le x \le 0) \\ 0 & \text{elsewhere.} \end{cases}$$

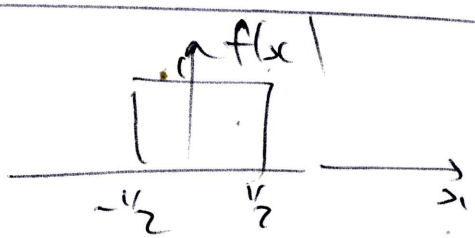
Q4 sheet 5

Q1a 1d

$$\boxed{\mathcal{F}[f(x)] = F(\omega) \Longrightarrow \mathcal{F}[F(x)] = 2\pi f(-\omega)}$$

~~Q1a~~

Q1a $\quad \mathcal{F}[f(x)] = \dfrac{\sin \frac{\omega}{2}}{\omega/2}$ $\quad$ F(ω)



$$\mathcal{F}\Big[\underbrace{\frac{\sin x/2}{x/2}}_{F(x)}\Big] = 2\pi f(-\omega)$$

$$= \begin{cases} 2\pi & (|\omega| \le 1/2) \\ 0 & (\text{otherwise}) \end{cases}$$

$\boxed{\text{See Q1}}$