Department of Mechanical Engineering, University of Bath

Modelling Techniques 2 ME20021

Notes on the technique of separation of variables

- When performing the technique of separation of variables on a partial differential equation with two independent variables, one always gets an ordinary differential equation to solve after factoring out a sine or cosine in the other variable. In most cases we obtain one of the three following equations:

$$y' + cy = 0, \qquad y'' - c^2 y = 0, \qquad y'' + c^2 y = 0. \tag{1a,b,c}$$

  These three equations arise from Fourier's equation, Laplace's equation and the wave equation, respectively. Their respective solutions are

$$y = Ae^{-ct}, \qquad y = Ae^{-ct} + Be^{ct}, \qquad y = A\cos ct + B\sin ct. \tag{2a,b,c}$$

  It is vitally important that you remember these solutions, and that they are remembered in a way which is independent of the notation used. For example, the equation

$$T'' + n^2 c^2 \pi^2 T = 0, \tag{3}$$

  is identical in form to (1c) and its solutions is

$$T = A\cos nc\pi t + B\sin nc\pi t. \tag{4}$$

- There is nothing magic about whether one uses A or B or any other letter as the arbitrary constant. It will be frequently the case that the 'wrong' one will be chosen, in the sense that one might end up with a Fourier Sine Series with A_n as the Fourier coefficients, when they are traditionally called B_n. At that stage in the analysis the coefficients are still arbitrary, and you can change the notation.

- If the solutions given in (2) have been forgotten, then always has the fallback position of using the substitution $y = \exp \lambda t$. In the three respective cases this subsitution yields,

$$\lambda = -c, \qquad \lambda = \pm c, \qquad \lambda = \pm cj, \tag{4a,b,c}$$

  which are to be compared with the solutions shown in (2).

- Finally, an additional means by which the exponentials versus sinusoids decision is made arises from the fact that wave equation describes wave motion, and therefore once should expect sines and cosines in time. One doesn't expect a violin string or drumskin to explode exponentially once disturbed.