Department of Mechanical Engineering, University of Bath

Modelling Techniques S2 ME20021 Sheet 1

Separation of Variables for PDEs: Fundamental Solutions

These questions are designed to give you practice in finding suitable fundamental solutions of the various PDEs we will be covering in my part of this unit. The questions are incomplete in the sense that the application of the very last boundary/initial condition is missing which will require the application of a Fourier Series formula.

Q1. The aim is to solve Laplace's equation in an infinite strip.

(i) Find the particular solution of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the domain $0 \leq x \leq 1$, $0 \leq y \leq \infty$, which satisfies $u = 0$ on $x = 0$ and $x = 1$ and which decays as $y \to \infty$. Hence write down a Fourier Sine Series solution by superposing all such solutions obtained for positive integer values of n. If $u = f(x)$ at $y = 0$, then find a Fourier Sine Series expression for $f(x)$. [Note: this follows the lecture notes precisely, but do try it out first without reference to the notes.]

(ii) How does the solution found in part (i) need to change if one were to solve Laplace's equation in the domain, $0 \leq x \leq d$, $0 \leq y \leq \infty$, where $u = 0$ on both $x = 0$ and $x = d$?

(iii) [A bit awkward.] How does this solution change if the domain is now $-d \leq x \leq d$ with $u = 0$ on both of these boundaries?

Q2. Solve Fourier's equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

in the domain $0 \leq x \leq 1$, $0 \leq t \leq \infty$, which satisfies the boundary conditions $u = 0$ at both $x = 0$ and $x = 1$. Hence write down a Fourier Sine Series solution by superposing all such solutions. If $u = f(x)$ at $t = 0$, then find a Fourier Sine Series expression for $f(x)$.

Determine how the above changes when the spatial domain is $0 \leq x \leq d$ and when $-d \leq x \leq d$, as in Q1.

Q3. Solve the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions, $u = 0$ on $x = 0$ and $x = 1$, and write down the most general possible solution in terms of a Fourier Sine series.

(i) Apply the initial conditions, $u = f(x)$ and $\partial u/\partial t = 0$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series.

(ii) Apply the initial conditions, $u = 0$ and $\partial u/\partial t = f(x)$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series. What is the physical interpretation of this pair of initial conditions?

Q4. Vibrations of a taut string are often slightly damped, and therefore an extra damping term will be present in the wave equation:

$$\frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}.$$

Solve this equation in the domain $0 \leq x \leq 1$, $0 \leq t \leq \infty$, where u satisfies the boundary conditions $u = 0$ at both $x = 0$ and $x = 1$. Hence write down a Fourier Sine Series solution by superposing all such solutions.

[Note that we are assuming light damping which, in the present context, means that $k < \pi c$. You will find it easier to write things down by introducing the quantity, d_n, where $d_n = \sqrt{n^2c^2\pi^2 - k^2}$.]

Q5. The aim is to solve the wave equation for beams,

$$\frac{\partial^2 u}{\partial t^2} + c^4\frac{\partial^4 u}{\partial x^4} = 0,$$

which is used for relatively thick vibrating structures. Analytical progress may be made with pin-jointed beams for which the boundary conditions are that both $u = 0$ and $\partial^2 u/\partial x^2 =$ on $x = 0$ and $x = 1$.

Write down the most general possible solution in terms of a Fourier Sine series. Apply the initial conditions, $u = f(x)$ and $\partial u/\partial t = 0$ at $t = 0$, and find an expression for $f(x)$ as a Fourier Sine Series.

Q6. Solve Fourier's equation,

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}.$$

in the domain $0 \leq x \leq 1$, $0 \leq t \leq \infty$, which satisfies the boundary conditions $\partial u/\partial x = 0$ at both $x = 0$ and $x = 1$. (Note that sines do not satisfy these boundary conditions, and therefore you'll need to use something else....)