Department of Mechanical Engineering, University of Bath

S2 Modelling Techniques Sheet 5

Fourier Transforms: Introductory Parts

Q1. Sketch the following functions and find their Fourier Transforms:
(a) $f(x) = 1,\ (|x| < 0.5), \quad f(x) = 0$, otherwise.
(b) $f(x) = 1,\ (0 < x < 1), \quad f(x) = 0$, otherwise.
(c) $f(x) = 1 - x,\ (0 \le x \le 1), \quad f(x) = 1 + x,\ (-1 \le x \le 0), \quad f(x) = 0$, otherwise.
(d) $f(x) = \sin x,\ (-\pi \le x \le \pi), \quad f(x) = 0$, otherwise.
(e) $f(x) = e^{-x},\ (-1 < x < 1), \quad f(x) = 0$, otherwise.
(f) $f(x) = -1,\ (-1 < x < 0), \quad f(x) = +1,\ (0 < x < 1), \quad f(x) = 0,$ (otherwise).

In all cases, take advantage of any symmetry in the given expression for $f(x)$ to simplify the Fourier Transform integral.

Q2. A unit pulse is defined as being, $P(x) = 1$ when $-\frac{1}{2} < x < \frac{1}{2}$, and it is zero otherwise. Find the convolution, $P(x) * P(x)$. It will be worth sketching both $P(\xi)$ and $P(x - \xi)$ in order to determine where they overlap.

Q3. (A little difficult!) Find the Fourier transform of $f(x)$ where $f(x) = e^{-x}$ for $x > 0$ and $f(x) = 0$ for $x < 0$; this is not the same as Q1e, above.

Use the result that the Fourier transform of the convolution of two functions is the product of their respective transforms (the convolution theorem) to show that the inverse transform of $1/(1 + \omega j)^2$ is a function which is equal to xe^{-x} for $x > 0$ and equal to zero for $x < 0$. (Hint: you will need to be particularly careful about where the functions in the convolution integral are nonzero, and to take this into account when modifying the limits of integration.)

Q4. Use some suitable Fourier Transforms from Q1 together with the Symmetry Theorem to find the Fourier Transforms of the following functions:

$$\text{(i)}\ \frac{\sin x/2}{x/2}; \qquad \text{(ii)}\ \frac{\sin \pi x}{x^2 - 1}; \qquad \text{(iii)}\ \frac{1 - \cos x}{x}; \qquad \text{(iv)}\ \frac{\sin x}{x}.$$

In the last case you will need to find the function of x whose FT is $(\sin \omega)/\omega$.

Q5. Suppose $f(x)$ represents a transmitted signal. This signal is *frequency modulated* when it is multiplied by a sinusoidal signal of frequency ω_c. Use the definition of the Fourier transform to prove the Frequency Modulation theorem:

$$\mathcal{F}[f(x) \cos \omega_c x] = \Big[F(\omega + \omega_c) + F(\omega - \omega_c)\Big]/2.$$

Q6. [Note that this is a time-dependent example, not an x-dependent one, but it may be solved in exactly the same way.]

The function $f(t)$ is equal to $e^{-at} \sin bt$ when $t > 0$, and is zero otherwise. Show that the Fourier Transform of $f(t)$ is given by

$$\mathcal{F}[f(t)] = \frac{b}{(\omega j)^2 + 2a(\omega j) + (a^2 + b^2)}.$$

Use all the necessary results given in the lecture notes to show that the following forced mass spring damper system,

$$y'' + 2ay' + (a^2 + b^2)y = g(t),$$

where both a and b are positive, has a solution which can be written in the form

$$y(t) = \frac{1}{b} \int_0^\infty g(t - \tau) e^{-a\tau} \sin b\tau \, d\tau.$$

If the forcing function were $g(t) = \delta(t)$, the unit impulse at $t = 0$, then what is the solution?

Q7. Write down the definition of the Fourier transform of $f(x)$ and differentiate it twice with respect to ω to obtain

$$\mathcal{F}[x^2 f(x)] = -\frac{d^2 F}{d\omega^2}.$$

Use this result and the time differentiation result given in the lecture notes to obtain the Fourier transform of the equation

$$\frac{d^2 f}{dx^2} - (1 + x^2) f = 0.$$

Would you use Fourier transforms to solve this problem?

Q8. This question will involve solving another time-dependent equation, and so the notation (t) will be slightly different from in the lecture notes (x). You will also need to assemble some armoury before solving the ODE at the end. Two of these are the Fourier Transforms of both $\delta(t - a)$ and $H(t)e^{-t}$. The other two are the formulae (i) for the FT of a single derivative and (ii) for what was called the x-shift theorem in the lectures, but which will now called the t-shift theorem for this question. Finally, we need to define the following function,

$$Ш(t) = \sum_{n=-\infty}^{\infty} \delta(t - n).$$

This is the Shah function which is named after the Cyrillic letter, sha, which it resembles. It is also called the Dirac comb, the bed-of-nails function or, somewhat unimaginatively, as a train of unit impulses. No doubt you can now imagine it!

The objective is solve the ODE,

$$y' + y = Ш(t)$$

using Fourier Transforms. Begin by finding the FT of the Shah function using its definition, and then find the FT of the ODE, eventually solving for $Y(\omega)$, which is the FT of $y(t)$. So $Y(\omega)$ should be in the form of an infinite sum, the inverse FT of which may be found using the t-shift theorem.

What does the solution, $y(t)$, look like?