ON THE TRULY MESHLESS SOLUTION OF HEAT CONDUCTION PROBLEMS IN HETEROGENEOUS MEDIA

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A truly meshless method based on the weighted least-squares (WLS) approximation and the method of point collocation is proposed to solve heat conduction problems in heterogeneous media. It is shown that, in the case of strong heterogeneity, accurate and smooth solutions for temperature and heat flux can be obtained by applying the WLS approximation in each homogeneous domain and using a double-stage WLS approximation technique together with a proper neighbor selection criterion at each interface.

1. INTRODUCTION

In recent years, a large family of meshless methods with the aim of getting rid of mesh constraints has been developed for solving partial differential equations. The basic idea of meshless methods is to provide numerical solutions on a set of arbitrarily distributed points without using any mesh to connect them. Compared to mesh generation, it is relatively simple to establish a point distribution and adapt it locally. The points are grouped together in “clouds” for which a local approximation for the problem variables is written. Depending on the methodology used to discretize the equations, meshless methods can be classified into two major categories: meshless strong-form methods and meshless weak-form methods. Most meshless weak-form methods, such as the element-free Galerkin method [1], are “meshless” only in terms of the numerical approximation of field variables, and they have to use a background mesh to do numerical integration of a weak form over the problem domain, which is computationally expensive. Meshless strong-form methods, such as the generalized finite-difference method [2], often use the point collocation method to satisfy governing partial differential equations and boundary conditions. They are simple to implement and computationally efficient. Since they do not need any background mesh, they are truly meshless methods.

The finite point method (FPM) is a truly meshless method proposed by Oñate et al. [3]. The FPM uses the weighted least-squares (WLS) approximation within each
NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>vector of unknowns</td>
</tr>
<tr>
<td>( A )</td>
<td>matrix of computation</td>
</tr>
<tr>
<td>( b )</td>
<td>vector of function differences</td>
</tr>
<tr>
<td>( C )</td>
<td>matrix of truncation errors</td>
</tr>
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<td>( e )</td>
<td>scalar field</td>
</tr>
<tr>
<td>( f )</td>
<td>matrix of final algebra equations</td>
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<td>( h )</td>
<td>searching radius</td>
</tr>
<tr>
<td>( m )</td>
<td>number of neighbor points</td>
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<tr>
<td>( n )</td>
<td>normal coordinate</td>
</tr>
<tr>
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</tr>
<tr>
<td>( p )</td>
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</tr>
<tr>
<td>( g )</td>
<td>normal heat flux</td>
</tr>
<tr>
<td>( Q^x, Q^y )</td>
<td>components of heat flux vector</td>
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<tr>
<td>( s )</td>
<td>heat source</td>
</tr>
<tr>
<td>( T )</td>
<td>temperature</td>
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<tr>
<td>( w )</td>
<td>weight function</td>
</tr>
<tr>
<td>( x )</td>
<td>position vector</td>
</tr>
<tr>
<td>( x, y )</td>
<td>Cartesian coordinates</td>
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<td>( \kappa )</td>
<td>thermal conductivity</td>
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The present authors revisited the method of Sadat et al. [14] for heterogeneous heat conduction problems and found that fluctuations existed in the predicted results of temperature and heat flux in the case of strong heterogeneity. In this article we report this finding and show that accurate and smooth solutions for temperature and heat flux can be obtained by applying the usual WLS approximation technique in each homogeneous domain and using the double-stage WLS approximation technique together with a proper neighbor selection criterion at each interface.

The article is organized as follows. In Section 2 the weighted least-squares approximation method is briefly described. Then, the double-stage WLS approximation technique and the proposed method for solving heat conduction problems in heterogeneous media are presented. Numerical examples are considered in Section 4 for the purpose of evaluating accuracy of different methods. The article ends with concluding remarks in Section 5.
2. THE WEIGHTED LEAST-SQUARES METHOD

The weighted least-squares method gives an approximation of a function \( f(x) \) and its derivatives at a given point \( x \) by using only the discrete function values at the neighbor points in the support domain of \( x \) (usually a ball in three dimensions or a disk in two dimensions). An advantage of this method is that it does not require regular distribution of points. In the following, we briefly explain the method in two dimensions (extension to three dimensions is straightforward).

Consider a Taylor expansion of \( f(x_i) \) around \( x \),

\[
f(x_i) = f(x) + \sum_{a=1}^{2} f_a(x)(x_{ia} - x_a) + \frac{1}{2} \sum_{a,b=1}^{2} f_{ab}(x)(x_{ia} - x_a)(x_{ib} - x_b) + e_i \tag{1}
\]

where \( e_i \) is the truncation error in the Taylor series expansion (here only to second-order; higher-order expansions are, of course, possible), \( f_a \) is the derivative with respect to \( x_a \) (the \( a \)th component of the position vector \( x \)), and \( f_{ab} \) is the derivative with respect to \( x_a \) and \( x_b \). The symbols \( x_{ia} \) and \( x_{ib} \) denote the \( a \)th and \( b \)th components of the position vector \( x_i \), respectively. From the given function values \( f(x) \) and \( f(x_i) \) \( (i = 1, 2, \ldots, m) \), the unknowns \( f_a \) and \( f_{ab} \) for \( a, b = 1, 2 \) (note that \( f_{ab} = f_{ba} \)) are computed by minimizing the error \( e_i \) for \( i = 1, 2, \ldots, m \). Here \( m \) is the number of neighbor points inside the support domain of \( x \).

Using the Taylor expansion (1) repeatedly for \( i = 1, 2, \ldots, m \), the system of equations for the five unknowns can be written as

\[
e = Ma - b \tag{2}
\]

with

\[
e = [e_1, e_2, \ldots, e_m]^T
\]

\[
a = [f_1, f_2, f_{11}, f_{12}, f_{22}]^T
\]

\[
b = [f(x_1) - f(x), f(x_2) - f(x), \ldots, f(x_m) - f(x)]^T
\]

\[
M = [p_1, p_2, \ldots, p_m]^T
\]

where \( a \) is the vector containing the five unknowns and \( M \) is a matrix in which the vector \( p_i \) is defined as

\[
p_i = \left[ x_{i1} - x_1, x_{i2} - x_2, \frac{(x_{i1} - x_1)^2}{2}, (x_{i1} - x_1)(x_{i2} - x_2), \frac{(x_{i2} - x_2)^2}{2} \right]^T \tag{3}
\]

For \( m > 5 \), this system is overdetermined with respect to the five unknowns in \( a \). This problem can be simply overcome by determining the unknown vector \( a \) by
minimizing the quadratic form

\[ J = \sum_{i=1}^{m} w_i e_i^2 \]  

(4)

where \( w_i = w(x_i - x) \) is the value of a weight function \( w \) at point \( x_i \). Standard minimization of \( J \) with respect to \( a \) gives

\[ a = C^{-1}Ab \]  

(5)

where

\[ C = \sum_{i=1}^{m} w_ip_ip_i^T \]  

(6)

\[ A = [w_1p_1, w_2p_2, \ldots, w_mp_m] \]  

(7)

In this article, we use a Gaussian weight function of the form

\[ w(r, h) = \begin{cases} \exp\left(-\epsilon r^2/h^2\right) & \text{if } r \leq h \\ 0 & \text{else} \end{cases} \]  

(8)

where \( r = \|x_i - x\| \) and \( \epsilon \) is a positive constant chosen to be equal to 6.3 in our computations. The size of the searching radius \( h \) determines \( m \), the number of neighboring points around \( x \) to be used for WLS approximation.

3. STEADY HEAT CONDUCTION IN A HETEROGENEOUS MEDIUM AND NUMERICAL METHODS

Two-dimensional steady heat conduction in a heterogeneous medium is governed by the Laplace equation as follows:

\[ \frac{\partial}{\partial x}\left( \kappa \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left( \kappa \frac{\partial T}{\partial y} \right) = s \]  

(9)

where \( T \) is the temperature, \( \kappa \) is the thermal conductivity of the medium, and \( s \) is the source term. On the boundaries, Dirichlet boundary condition,

\[ T - T_b = 0 \quad \text{on } \Gamma_d \]  

(10)

and Neumann boundary condition,

\[ \kappa \frac{\partial T}{\partial n} - q = 0 \quad \text{on } \Gamma_n \]  

(11)

are to be satisfied with the prescribed values of the temperature \( T_b \) and the normal heat flux \( q \).
To obtain the discretized equations, the point collocation method is applied, i.e., the Laplace equation (9) and the boundary conditions (10)–(11) are forced to be satisfied at each internal and boundary point, respectively. This gives the set of equations

\[ [L]_i = \left[ \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x} \right) \right]_i + \left[ \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) \right]_i - s_i = 0 \quad \text{in } \Omega \] (12)

\[ T_j - T_b^j = 0 \quad \text{on } \Gamma_d \] (13)

\[ \kappa \frac{\partial T}{\partial n} - q_k = 0 \quad \text{on } \Gamma_n \] (14)

Expressing spatial derivatives occurring in the above set of equations in terms of the unknown temperatures at points by the WLS method leads to the final discretized system of equations

\[ \mathbf{Gh} = \mathbf{f} \] (15)

where \( \mathbf{G} \) is the coefficient matrix, vector \( \mathbf{h} \) contains the unknown temperatures, and \( \mathbf{f} \) is a vector containing the contributions from the prescribed values \( s_i, T_b^j, \) and \( q_k \).

How to express the two derivative terms in \( [L]_i \) in terms of the unknown temperatures is an important issue. Taking the first term as an example, it can be expanded as

\[ \left[ \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x} \right) \right]_i = \sum_{j=1}^{N} a_{ij} Q^x_j \] (17)

where

\[ Q^x_j = \left[ \kappa \frac{\partial T}{\partial x} \right]_j \] (18)
and $a_{ij}$ are the coefficients in the WLS approximation of the first derivative with respect to $x$ at point $i$. Here $N$ is the total number of points. Note that only the coefficients corresponding to the neighboring points of $i$ are nonzero. The term $Q_j^x$ is determined by

$$Q_j^x = \kappa_j \sum_{n=1}^{N} a_{jn} T_n$$  \hspace{1cm} (19)

where $\kappa_j$ corresponds to the thermal conductivity at point $j$ and $a_{jn}$ are the coefficients in the WLS approximation of the first derivative with respect to $x$ at point $j$. The discretization of the first term in $[L]_i$ is finally written as

$$\sum_{j=1}^{N} \sum_{n=1}^{N} \kappa_j a_{ij} a_{jn} T_n$$  \hspace{1cm} (20)

This expression takes into account the neighbors of the neighbors of the point considered. For a point on the interface separating two domains of different thermal conductivities, Sadat et al. [14] suggested that its thermal conductivity can be taken from one of the two domains. However, in the next section, we will show that different choice of thermal conductivity gives different results.

From our numerical tests, we found that the method of Sadat et al. using the double-stage WLS approximation described above yields much improved results. However, the simulated results exhibited prominent fluctuations in both temperature and heat flux. One reason is that, in heterogeneous heat conduction problems, the first derivatives of the temperature are discontinuous at the interface separating two domains of different thermal conductivities, so the WLS approximation of the first derivatives of the temperature for the points near the interface introduces large errors when the neighboring points are chosen as usual from both domains. A proper neighbor selection criterion was proposed to solve the problem. That is, for a given point, it only selects its neighboring points from among those that have the same thermal conductivity and those that are located on the interface. This criterion is only applied at the first stage, where the first derivatives of the temperature are approximated by WLS. Although introducing the criterion into the method of Sadat et al. improved the results in one-dimensional problems, it still produced fluctuations in the results of two-dimensional problems. To remove the fluctuations completely, we propose a new method as follows. For a point belonging to a homogeneous domain, Eq. (12) can be simplified as

$$[L]_i = \kappa \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] - s_i = 0$$  \hspace{1cm} (21)

The second-order derivatives in Eq. (21) are approximated by WLS together with the proper neighbor selection criterion introduced above. For a point on the interface, the double-stage WLS approximation is used to discretize Eq. (12). Again, to calculate heat fluxes at the first stage, the proper neighbor selection criterion is applied.
Heat flux calculated on an interface point can have different values corresponding to different choices of the thermal conductivity value for that point. In order to obtain a unique solution, the averaged heat flux is adopted in the proposed method.

Hereafter, we will call the meshless point collocation method proposed by Sadat et al. [14] the MPCM1 and the new method proposed in this article the MPCM2.

4. NUMERICAL TESTS

To evaluate the two methods, MPCM1 and MPCM2, a steady-state heat conduction problem in a two-layered composite wall, as depicted in Figure 1, is first solved in this section. This test problem was proposed by Sadat et al. [14] to validate their method, i.e., MPCM1 here. The two layers have the same thickness but different conductivities of $\kappa_1$ and $\kappa_2$. The ratio of conductivities $\kappa_1/\kappa_2$ accounts for the degree of heterogeneity of the wall. Prescribed temperatures are imposed on the two vertical surfaces, whereas the horizontal boundaries are assumed to be adiabatic. The computational domain was considered to be a square with dimensionless length 1. A total of 441 (21 x 21) points were first evenly placed at equal distances ($\Delta x = \Delta y = 0.05$) over the problem domain, with 80 of them located right on the four sides of the square (21 on each side), which are so-called boundary points, and 21 of them located right on the interface between the two layers, which are so-called interface points. Then, the positions of boundary and interface points were fixed while the positions of other points were shifted randomly to give an irregular point distribution, as shown in Figure 2. The searching radius $h$ was set to 2.1$\Delta x$. Both the case with a source term ($s \neq 0$) and the one without a source term ($s = 0$) are considered. The stationary temperature and heat flux profiles along the $x$ direction obtained by the MPCM1 and MPCM2 are compared to the exact

Figure 1. Two-dimensional heat conduction in a two-layered composite wall.
analytical solutions. The parameter values and the results are given in dimensionless form.

Figures 3 and 4 show the results of temperature and heat flux for $k_1/k_2 = 0.01$ with $s = 0$ and $k_2$ fixed to 1. It can be seen that the MPCM1 provides acceptable results for the heat conduction problem in the case of strong heterogeneity.

![Figure 2. Point distribution used for space discretization.](image)

![Figure 3. Temperature profile for $k_1/k_2 = 0.01$ and $s = 0$. Here MPCM1 (k1) and MPCM1 (k2) mean the thermal conductivities on the interface points are set to $k_1$ and $k_2$, respectively. The small figure inside is an enlarged plot for the region $0.5 \leq x \leq 1$.](image)
Nevertheless, the results predicted by the MPCM1, especially the heat flux results, exhibit prominent fluctuations, as evidenced in Figure 4. Moreover, the results of the MPCM1 are sensitive to the choice of the thermal conductivities on the interface points. These problems are solved perfectly by the MPCM2, and the results obtained by the MPCM2 match the exact solutions accurately. In Figures 5 and 6, we show the results obtained by the MPCM1 using the same parameters but a finer and

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Heat flux profile for $\kappa_1/\kappa_2 = 0.01$ and $s = 0$. Here MPCM1 (k1) and MPCM1 (k2) mean the thermal conductivities on the interface points are set to $k_1$ and $k_2$, respectively.

![Figure 5](https://example.com/figure5.png)

**Figure 5.** Temperature profile for $\kappa_1/\kappa_2 = 0.01$ and $s = 0$, predicted by the MPCM1 using two different grids.
regular grid of 1,681 (41 × 41) points. Oscillations are still observed, but they become smaller.

For the case of $s = -10$, the results with $\kappa_1 / \kappa_2 = 0.01$ are shown in Figures 7 and 8 for temperature and heat flux, respectively. Again, the MPCM1 yields fluctuating results for both temperature and heat flux, while the results of the MPCM2 are fluctuation-free and are very close to the analytical solutions.
The test problem solved above is actually one-dimensional. To further test the MPCM1 and MPCM2, we here consider a two-dimensional problem, as shown in Figure 9. The whole domain is a square with dimensionless length 1. The center part of the domain is a square with dimensionless length 0.5 and is made of a material ($\kappa_1 = 0.01$) different from that of the outer part ($\kappa_2 = 1$). A regular grid with a total of 1,681 ($41 \times 41$) points is used in the simulation. Shown in Figures 10 and 11 are the temperature profiles along a horizontal line and a vertical line, respectively.

![Figure 8. Heat flux profile for $\kappa_1/\kappa_2 = 0.01$ and $s = -10$.](image)

**Figure 8.** Heat flux profile for $\kappa_1/\kappa_2 = 0.01$ and $s = -10$.

**Figure 9.** Two-dimensional heat conduction in a square domain with the center square made of a material different from that of the outer part.
Results from a finite-element method are used for comparison. Again, the MPCM2 outperforms the MPCM1 in terms of accuracy and smoothness.

5. CONCLUSIONS

A truly meshless method based on the WLS approximation and the point collocation approach has been presented for the numerical simulation of heat

![Figure 10. Temperature profile along the horizontal line at y = 0.5 for $\kappa_1/\kappa_2 = 0.01$ and $s = 0$.](image1)

![Figure 11. Temperature profile along the vertical line at $x = 0.875$ for $\kappa_1/\kappa_2 = 0.01$ and $s = 0$.](image2)
conduction problems in heterogeneous media. The proposed method employs the WLS approximation for points inside the homogeneous domains and the double-stage WLS approximation for points at interfaces. In addition, the proper neighbor selection criterion is introduced for the double-stage WLS approximation. It has been shown that the proposed method is able to obtain accurate and smooth solutions in the case of strong heterogeneity. The proposed method can be easily extended to transient problems.

REFERENCES