According to Abramowitz & Stegun (II)

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Abstract
This paper deals with further OpenMath content dictionaries meant to capture the properties of the transcendental functions (see [5] for the first steps in this direction). In particular we introduce transc3.ocd containing the multi-valued definitions of the inverse functions.

1 Introduction
We saw in [5] that it is necessary in practice to be very careful with what is known in theory to all mathematicians, that to produce single-valued functions $C \rightarrow C$ (or $C \setminus \{\text{singularities}\} \rightarrow C$) which are the inverses of functions such as exp, it is necessary to introduce branch cuts [10]. It then follows that many identities that one might hope to be true are in fact not true with a particular choice of branch cuts, or indeed with any consistent choice of branch cuts. [8] shows two errors in [2] caused by confusion over the precise handling of the branch cut for log.

[5] and the OpenMath [1, 9] transc1 Content Dictionary define, essentially in agreement with [2], a set of such single-valued functions. However, this does not easily express all the issues involved with such inverse functions.

2 arctan and arg
It is common to say, or at least believe, that, for real $x$ and $y$,

$$\arg(x + iy) = \arctan \left( \frac{y}{x} \right).$$

\[ \text{Equation (1)} \]

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but a moment’s consideration of ranges (a tool that we have found very valuable in this area) shows that it cannot be so: the left-hand side has a range of \((-\pi, \pi]\) with the standard branch cuts, and certainly has a range of size $2\pi$, whereas the right-hand side has a range of size $\pi$.

The fundamental problem is, of course, that considering $\frac{y}{x}$ immediately confuses $1+i$ with $-1-i$. This fact was well-known to the early designers of FORTRAN, who defined a two-argument function $\text{ATAN2}$, such that

$$\text{ATAN2}(y, x) = \arctan\left(\frac{y}{x}\right) \pm \pi.$$  

(2)

More precisely, the correction factor is 0 when $x > 0$, $+\pi$ when $x < 0$ and $y \geq 0$, and $-\pi$ when $x, y < 0$. For completeness, one should also define what happens when $x = 0$, when the answer is $+\pi/2$ when $y > 0$ and $-\pi/2$ when $y < 0$. This function is also present in the computer algebra system Maxima, with the somewhat gnomic definition: $\text{ATAN2}(y, x)$ yields the value of $\text{ATAN}(y/x)$ in the interval $-\pi$ to $\pi$.

This has been added to OpenMath, as the symbol $\text{arctan}$ in the $\text{transc2}$ Content Dictionary. Use of this enables us to rescue the incorrect equation insert from MathComp.

3 The Unwinding Number

In [5], we found the following concept very helpful in stating globally-valid equations among the inverse functions: the unwinding number $K$, defined\(^1\) by

$$K(z) = \frac{z - \ln \exp z}{2\pi i} = \left\lceil \frac{3z - \pi}{2\pi} \right\rceil \in \mathbb{Z}. \tag{3}$$

Since OpenMath Content Dictionaries may well wish to state such equations (even only to stop people assuming, falsely, simplified versions of them), it seemed worth adding this to the $\text{transc2}$ CD, as the symbol $\text{unwind}$. The various statements in equation (3) translate easily into OpenMath, e.g. the second part becomes the following.

\[
<\text{OMA}>
<\text{OMS cd="relation1" name="eq"} />
<\text{OMA}>
    <\text{OMS cd="transc2" name="unwind"} />
    <\text{OMV name="z"} />
</\text{OMA}>
<\text{OMA}>
    <\text{OMS cd="rounding1" name="ceiling"} />
</\text{OMA}>
\]

\(^1\)Note that the sign convention here is the opposite to that of [7], which defined $K(z)$ as $\left\lfloor \frac{\pi - 3z}{2\pi} \right\rfloor$: the authors of [7] recanted later to keep the number of $-1$s occurring in formulae to a minimum.
The correct equation

\[ \arcsin z = \arctan \frac{z}{\sqrt{1 - z^2}} + \pi \mathcal{K}(\ln(1 + z)) - \pi \mathcal{K}(\ln(1 - z)) \]  

becomes an equally straight-forward piece of OpenMath.
4 Multi-valued functions

Mathematical texts often urge us\(^2\) to treat these one–many inverse functions as multi-valued, defining, say, \(\text{Arctan}(x) = \{y \mid \tan y = x\} = \{\text{arctan}(x) + n\pi \mid n \in \mathbb{Z}\}\) (the notational convention of using capital letters for these set-valued functions seems helpful). It should be noted that \(\text{Arctan}\) is deceptively simple in this respect, and the true rules for the inverse trigonometric functions are [2, (4.4.10–12)]

\[
\begin{align*}
\text{Arccsc}(z) & = \{(-1)^k \arcsin(z) + k\pi \mid k \in \mathbb{Z}\} \quad (10)
\end{align*}
\]

where we have changed to our set-theoretic notation, and added the last three equations, which are clearly implied by the first three.

In this interpretation,

\[
\text{Arctan}(x) + \text{Arctan}(y) \equiv \text{Arctan}\left(\frac{x + y}{1 - xy}\right),
\]

with the + on the left-hand side representing element-by-element addition of sets, is valid, whereas the single-valued

\[
\text{arctan}(x) + \text{arctan}(y) \equiv \text{arctan}\left(\frac{x + y}{1 - xy}\right)
\]

is false (e.g. \(x = y = 2\)).

Most of the "algebraic" rules of simplification are valid in this context, e.g. with \(\sqrt{x} = \{y \mid y^2 = x\} = \{\pm \sqrt{x}\}\), it is the case that

\[
\sqrt{x} \sqrt{y} = \sqrt{xy}
\]

(interpreting the multiplication as element-by-element multiplication of sets), whereas

\[
\sqrt{x} \sqrt{y} \neq \sqrt{xy}
\]

is not true: consider \(x = y = -1\).

These multivalued functions have been added to OpenMath in the \texttt{transc3} Content Dictionary\(^3\) with names such as \texttt{ln}. The definitions are relatively

\(^2\)And we have found this idea useful in [4, 3].

\(^3\)The original suggestion in [5] was to add symbols such as \texttt{Ln}. While this would be possible, such symbols could not easily be added to \texttt{transc1}, since that is intended to reflect the definitions in MathML [11], and, while that does not have precise semantics, it seems that the default semantics of MathML’s \texttt{<ln/>} is single-valued. It is easier to attach different semantics, using the MathML \texttt{csymbol} construct, if the MathML and OpenMath symbols have the same name.
straight-forward in OpenMath, for example the following definition of $\ln$, which in symbols reads $\text{Ln}(x) = \{ y \mid \exp(y) = x \}$.

\[
\text{Ln}(x) = \{ y \mid \exp(y) = x \}
\]

It is also possible to state the relationship with the single-valued $\ln$ from $\text{transc1}$:

\[
\text{Ln}(x) = \{ \ln(x) + 2\pi i : n \in \mathbb{Z} \}
\]
\[(4.4.26) \quad \text{Arcsin } x = -i \ln \left( 1 - x^2 \right)^{1/2} + ix \quad x^2 \leq 1
\]
\[(4.4.27) \quad \text{Arccos } x = -i \ln \left( x + i(1 - x^2)^{1/2} \right) \quad x^2 \leq 1
\]
\[(4.4.28) \quad \text{Arctan } x = \frac{i}{2} \ln \frac{1 - ix}{1 + ix} = \frac{i}{2} \ln \frac{1 + x^2}{1 - x^2} \quad x \text{ real}
\]
\[(4.4.29) \quad \text{Arccsc } x = -i \ln \left( \frac{x^2 - 1}{x} \right)^{1/2} + ix \quad x^2 \geq 1
\]
\[(4.4.30) \quad \text{Arccsc } x = -i \ln \left( 1 + i \frac{(x^2 - 1)^{1/2}}{x} \right) \quad x^2 \geq 1
\]
\[(4.4.31) \quad \text{Arccot } x = \frac{i}{2} \ln \frac{1 + 1/x}{1 - 1/x} = \frac{i}{2} \ln \frac{x - 1}{x + 1} \quad x \text{ real}
\]

\[\text{Relationship with Ln}\]

It is well-known that the single-valued inverse trigonometric functions have expressions in terms of ln. \([2, (4.4.26-31)]\) give equivalent multivalued expressions in terms of Ln, as follows (we have preserved their notation). (4.4.28) and (4.4.31) are straight-forward enough, since they are simple translations of the singlevalued case. The indeterminacy of Ln is in multiplies of $2\pi i$, so this translates into the correct indeterminacy for Arctan, in multiplies of $\pi$. There is, in fact, no reason for the restriction to real $x$ in this case.
However, translating the other single-valued expression in terms of \( \ln \), i.e.

\[
\arcsin(z) = -i \ln \left( \sqrt{1 - z^2} + iz \right)
\]  

(14)

to multi-valued becomes more complex: in this case we have to read (4.4.26) as

\[
\text{Arcsin}(z) = -i \text{Ln} \left( \sqrt{1 - z^2} + iz \right).
\]  

(15)

Since \( \text{Sqrt} \) has two values (except when \( z = \pm 1 \)), this gives us two values in each interval of length \( 2\pi \), and the indeterminacy of \( \text{Ln} \) translates into the correct \( 2\pi \) general indeterminacy. But are these the right two values? If we accept the restriction of (4.4.26), then there is a fairly clear argument that \( r - i \ln \left( -\sqrt{1 - z^2} + iz \right) \) represents the correct “other value” to \( -i \ln \left( \sqrt{1 - z^2} + iz \right) \).

The input to \( \ln \) is a complex number of magnitude 1, so the value of \( \ln \) will be purely imaginary (which is just as well!). The effect of changing the sign on the square root can be regarded as negating the whole input, then taking the complex conjugate. Except on the branch cut, the effect of these operations is to add/subtract \( i\pi \) to the value, then negate the imaginary part. After the multiplication by \( -i \), the overall effect is to negate and add/subtract \( \pi \), as required.

The above proof does not translate easily into more general \( z \), but instead we can proceed as in [4, 6], and consider the regions of validity of

\[
\pi - \arcsin(z) = -i \ln \left( -\sqrt{1 - z^2} + iz \right).
\]  

(16)

The equation differentiates to

\[
-\frac{1}{\sqrt{1 - z^2}} = -\frac{1}{\sqrt{1 - z^2}},
\]

and is therefore “true up to a constant”, where the constant might be different in different regions of the complex plane. At \( z = 0 \), the left-hand side of equation (16) evaluates to \( \pi \), whereas the right-hand side evaluates to \( -i \ln(-1) = \pi \), so equation (16) is valid here. Now, the branch cuts\(^5\) of \( \arcsin(z) \) are \((-\infty, -1)\) and \((1, \infty)\), which also happen to be the branch cuts of \( \sqrt{1 - z^2} \). Call these two lines \( L_1 \) and \( L_2 \) respectively. We must also consider the branch cut of \( \ln(-\sqrt{1 - z^2} + iz) \) itself. Now the branch cut of \( \ln \) is on the negative real axis, so we want to find those \( z \) for which \( -\sqrt{1 - z^2} + iz = t \in (-\infty, 0) \). This equates to \( -\sqrt{1 - z^2} = t - iz \), so a necessary condition is that \( 1 - z^2 = (t - iz)^2 \), which simplifies to \( t^2 + 2izt - 1 = 0 \). Writing \( z = x + iy \), we see that the only imaginary term is \( 2ixt \), so \( x \) has to be zero. We are then left with \( t^2 + 2ty - 1 \), which gives \( t = -y - \sqrt{y^2 + 1} \) (choosing the sign to make \( t \) negative). We therefore see

\(^4\)One is tempted to say “argument”, but this causes confusion when dealing with complex numbers.

\(^5\)Curiously, [2] does not prescribe the values of \( \arcsin \) on its branch cuts: we here follow [10] and [5] and define the values of \( \arcsin \) on the branch cuts to be those that follow from equation (14), and similarly \( \arccos \) from equation (34).
that the whole imaginary axis is a branch cut: call it \( L \). If we let \( C_- \) and \( C_+ \) denote the (strict) left-hand and right-hand half-planes respectively, we see that \( L, L_1 \) and \( L_2 \) between them partition the complex plane into two 2-dimensional regions: \( C_- \setminus L_1 \) and \( C_+ \setminus L_2 \), and three one-dimensional regions, the branch cuts themselves.

\( C_- \setminus L_1 \) Since this region intersects \((-1, 1)\) on which we know equation (16) is true, it must be true throughout the region. As a check, with \( z = 1 + i \), both \( \arcsin(z) \) and \( -i \ln \left( \sqrt{1 - z^2} + iz \right) \) evaluate to \( 0.666 \ldots + 1.061 \ldots i \), whereas \( \pi - \arcsin(z) \) and \( -i \ln \left( \sqrt{1 - z^2} + iz \right) \) evaluate to \( \pi - 0.666 \ldots + 1.061 \ldots i \).

\( C_- \setminus L_2 \) Similarly.

\( L \) Since \( 0 \in L \) and we have verified the equation there, we know it is true. The cynical might like another check, e.g. at \( z = i \), where \( \arcsin(z) = 0.881 \ldots i \), so the alternative value should be \( \pi - 0.881 \ldots i \). This is indeed the value of \( -i \ln \left( -\sqrt{1 - z^2} + iz \right) \), whereas \( -i \ln \left( \sqrt{1 - z^2} + iz \right) \) does indeed evaluate to \( \arcsin(i) \).

\( L_1 \) A typical evaluation here would be at \( z = -2 \). When evaluating on the branch cut, one has to be careful, since typically the branch cut is continuous with one side of itself, but not the other, so a trivial numeric error\(^6\) may have serious consequences. It is better to proceed at least semi-symbolically. \( \arcsin(-2) \) is, by definition, \( -i \ln(\sqrt{1 - 2^2} - 2i) = -i \ln((-2 + \sqrt{3})i) = -i(\frac{\pi}{2} + \ln(2 - \sqrt{3})) = -\frac{\pi}{2} - i \ln(2 - \sqrt{3}) \). Hence \( \pi - \arcsin(-2) = \frac{3\pi}{2} + i \ln(2 - \sqrt{3}) \), and \( -i \ln \left( -\sqrt{1 - z^2} + iz \right) \) evaluates to \( -i \ln((-\sqrt{3} - 2i) = -i(\frac{\pi}{2} - \ln(2 + \sqrt{3})) = \frac{\pi}{2} - i \ln(2 + \sqrt{3}) \), which is \( 2\pi \) less than the correct value, and is therefore one of the allowable values. So the identity is verified here, and therefore throughout \( L_1 \).

\( L_2 \) A typical evaluation here would be at \( z = 2 \). \( \arcsin(2) \) is, by definition, \( -i \ln(\sqrt{1 - 2^2} + 2i) = -i \ln((2 + \sqrt{3})i) = -i(\frac{\pi}{2} + \ln(2 + \sqrt{3})) = \frac{\pi}{2} - i \ln(2 + \sqrt{3}) \). Hence the alternative value is \( \pi \) minus this, i.e. \( \frac{\pi}{2} + i \ln(2 + \sqrt{3}) \). \( -i \ln \left( \sqrt{1 - z^2} + iz \right) \) evaluates to \( -i \ln(-\sqrt{3} + 2i) = -i(\frac{\pi}{2} + \ln(2 - \sqrt{3})) = \frac{\pi}{2} - i \ln(2 - \sqrt{3}) = \frac{\pi}{2} + i \ln(2 + \sqrt{3}) \), so the identity is verified.

Hence in fact equation (16) is valid throughout \( C \), and the stipulation in [2, (4.4.26)] is invalid provided one defines the values of \( \arcsin \) on the branch cuts to be those that follow from equation (14). The Appendix contains similar proofs of 4.4.27, 4.4.29 and 4.4.30.

\(^6\)As the author has discovered.
6 OpenMath definitions in terms of Ln

Equation (14) does not translate directly into OpenMath with the current set of Content Dictionaries, and has to be expressed as

\[ \text{Arcsin}(z) = -i \left( \ln \left( \sqrt{1 - z^2} + iz \right) \cup \ln \left( -\sqrt{1 - z^2} + iz \right) \right). \]  

(17)

The OpenMath expression is as follows.

\[
\text{<FMP>}
\text{<OMOBJ>}
\text{<OMA>}
\text{<OMS name="eq" cd="relation1"/>}
\text{<OMA>}
\text{<OMS name="arcsin" cd="transc3"/>}
\text{<OMV name="z"/>}
\text{</OMA>}
\text{<OMA>}
\text{<OMS name="map" cd="set1"/>}
\text{<OMBIND>}
\text{<OMS name="lambda" cd="fns1"/>}
\text{<OMBVAR>}
\text{<OMV name="y"/>}
\text{</OMBVAR>}
\text{<OMA>}
\text{<OMS name="times" cd="arith1"/>}
\text{<OMA>}
\text{<OMS name="unary_minus" cd="arith1"/>}
\text{<OMS name="i" cd="nums1"/>}
\text{</OMA>}
\text{<OMV name="y"/>}
\text{</OMA>}
\text{</OMBIND>}
\text{<OMA>}
\text{<OMS name="union" cd="set1"/>}
\text{<OMA>}
\text{<OMS name="ln" cd="transc3"/>}
\text{<OMA>}
\text{<OMS name="plus" cd="arith1"/>}
\text{<OMA>}
\text{<OMS name="root" cd="arith1"/>}
\text{<OMA>}
\text{<OMS name="minus" cd="arith1"/>}
\text{<OMS name="one" cd="alg1"/>}
\text{</OMA>}
\text{<OMV name="z"/>}
\text{</OMA>}
\text{</OMA>}
\text{</OMOBJ>}
\text{</FMP>}

10
\[ z^2 e^{\ln z^2} = \sqrt{z^2 - 1} \]
### 7 Handling set-valued functions

There are a few caveats about the manipulation of such set-valued functions that must be mentioned, though.

1. Cancellation is no longer trivial, since in principle $\text{Arctan}(x) - \text{Arctan}(x) = \{n\pi \mid n \in \mathbb{Z}\}$, rather than being zero.

2. Not all such multivalued functions have such simple expressions as

\[
\text{Arctan}(x) = \{\text{arctan}(x) + n\pi \mid n \in \mathbb{Z}\},
\]

For example, $\text{Arcsin}(x) = \{\text{arcsin}(x) + 2n\pi \mid n \in \mathbb{Z}\} \cup \{\pi - \text{arcsin}(x) + 2n\pi \mid n \in \mathbb{Z}\}$. This problem combines with the previous one, so that, if $A = \text{Arcsin}(x)$,

\[
A - A = \{2n\pi \mid n \in \mathbb{Z}\} \cup \{2\text{arcsin}(x) - \pi + 2n\pi \mid n \in \mathbb{Z}\}
\]

\[
\cup \{\pi - 2\text{arcsin}(x) + 2n\pi \mid n \in \mathbb{Z}\}
\]

\[
= \{2n\pi \mid n \in \mathbb{Z}\} + \{0, 2\text{arcsin}(x) - \pi, \pi - 2\text{arcsin}(x)\}.
\]

Note that this still depends on $x$, unlike the case of $\text{Arctan}(x) - \text{Arctan}(x)$.

3. However, some equations take on somewhat surprising forms in this context, e.g. the incorrect simplification of equation (4)

\[
\text{arcsin}(z) = \text{arctan}\left(\frac{z}{\sqrt{1 - z^2}}\right),
\]

(18)

does not simply become

\[
\text{Arcsin}(z) = \text{Arctan}\left(\frac{z}{\sqrt{1 - z^2}}\right),
\]

(19)

since the left-hand side has a period of $2\pi$, with two values in each period, whereas the right-hand side has a period of $\pi$. If one simply allows for the multi-valued nature of $\sqrt{}$, the equation becomes the correct

\[
\text{Arcsin}(z) \subset \text{Arctan}\left(\frac{z}{\sqrt{1 - z^2}}\right),
\]

(20)

and if we want an equality of sets, we have

\[
\text{Arcsin}(z) \cup \text{Arcsin}(-z) = \text{Arctan}\left(\frac{z}{\sqrt{1 - z^2}}\right),
\]

(21)

in which both sides take on four values in each $2\pi$ period. It is an open question to produce an alternative characterisation of just $\text{Arcsin}(z)$. 


4. The equation
\[ \log(z^2) = 2 \log(z) \]  
(22)
is not valid if we interpret \(2 \log(z)\) as \(\{2y \mid \exp(y) = z\}\), since this has an indeterminacy of \(4\pi i\), and the left-hand side has an indeterminacy of \(2\pi i\). Instead we need to interpret \(2 \log(z)\) as \(\log(z) + \log(z)\), and under this interpretation, equation (22) is true, as a specialisation of the correct equation
\[ \log(z_1 z_2) = \log(z_1) + \log(z_2). \]  
(23)

Because of point 1, we do not use the OpenMath symbol \texttt{minus} directly to represent the subtraction of sets, since software may deduce that \(A - A = 0\). Instead, we have augmented OpenMath (the Content Dictionary \texttt{set2}) with an operation \texttt{lift_binary} to lift a binary operation from elements of a set \(X\) to a binary operation between subsets of \(X\). Hence \(A - A\) would be represented as
\[
<\text{OMA}>
<\text{OMA}>
<\text{OMS name="lift_binary" cd="set2"}>
<\text{OMS name="minus" cd="arith1"}>
</OMA>
<\text{OMV name="A"}>
<\text{OMV name="A"}>
</OMA>
\]

8 Arithmetic Rules on Multivalued Functions

8.1 Arithmetic with Arctan and Arccot

[2, 4.4.34] reads
\[ \arctan(z_1) \pm \arctan(z_2) = \arctan \left( \frac{z_1 \pm z_2}{1 \mp z_1 z_2} \right). \]  
(24)

It is tempting to read this as shorthand for the two equations
\[ \arctan(z_1) + \arctan(z_2) = \arctan \left( \frac{z_1 + z_2}{1 - z_1 z_2} \right) \]  
(25)
and
\[ \arctan(z_1) - \arctan(z_2) = \arctan \left( \frac{z_1 - z_2}{1 + z_1 z_2} \right). \]  
(26)

Each of (25) and (26) is in fact true, as can be easily verified by conversion into Ln form. The single-valued versions of them are not true, but only true modulo \(\pi\), as discussed in [4]. The situation is identical with Arccot, and with the useful “hybrid” equation [2, 4.4.36]
\[ \arctan(z_1) \pm \text{arccot}(z_2) = \text{arccot} \left( \frac{z_2 \mp z_1}{z_1 z_2 \pm 1} \right) = \arctan \left( \frac{z_1 z_2 \pm 1}{z_2 \mp z_1} \right), \]  
(27)
which can also be split into its two components.
8.2 Arithmetic with Arcsin and Arccos

[2, 4.4.32] reads

\[ \text{Arcsin}(z_1) \pm \text{Arcsin}(z_2) = \text{Arcsin} \left( z_1 \sqrt{1 - z_2^2} \pm z_2 \sqrt{1 - z_1^2} \right). \]  

(28)

This poses two questions.

1. Can we separate this equation, as with (24)?

2. Should we read \( \sqrt{\cdot} \) as Sqrt? But, if so, what is the point\(^7\) of the \( \pm \) on the right-hand side, since \( \pm \text{Sqrt} \) is meaningless?

Throughout this discussion, we will work modulo \( 2\pi \), so that \( \text{Arcsin}(z) \) has two values (except when \( z = \pm 1 \)).

If we read \( \sqrt{\cdot} \) as \( \sqrt{\cdot} \), then there is a problem of mismatch of cardinality: the left hand side has eight (reading \( \pm \)) or four (reading +) values, whereas the right-hand side has four (reading \( \pm \)) or two (reading +).

It is probably simpler to consider these questions in terms of Arccos. Here we should first note a consequence of equation (5):

\[ \text{Arccos}(z) = -\text{Arccos}(z). \]  

(29)

[2, 4.4.33] reads

\[ \text{Arccos}(z_1) \pm \text{Arccos}(z_2) = \text{Arccos} \left( z_1 z_2 \mp \sqrt{(1 - z_1^2)(1 - z_2^2)} \right). \]  

(30)

In view of (29), the \( \pm \) could just as well be a +, and \( \mp \sqrt{\cdot} \) is precisely equivalent to Sqrt. Here there is no problem of cardinality: each side has four values modulo \( 2\pi \).

With this in mind, let us return to the case of Arcsin. We should note that there is an equivalent to equation (29):

\[ \text{Arcsin}(z) = \pi - \text{Arcsin}(z). \]  

(31)

With this in mind, suppose that \( \text{Arcsin}(z_1) + \text{Arcsin}(z_2) = \text{Arcsin}(z_3) \) (note that \( z_3 \) is actually a multi-valued object). Then \( \pi - \text{Arcsin}(z_1) + \text{Arcsin}(z_2) = \text{Arcsin}(z_3) \). Hence \( \pi - \text{Arcsin}(z_1) + \text{Arcsin}(z_2) = \pi - \text{Arcsin}(z_3) \), and, simplifying, \( \text{Arcsin}(z_1) - \text{Arcsin}(z_2) = \text{Arcsin}(z_3) \). In other words, the \( \pm \) on left-hand side of equation (28) is somewhat spurious. Not totally though, unlike the case of equation (30), we have not proved that \( \text{Arcsin}(z_1) + \text{Arcsin}(z_2) = \text{Arcsin}(z_1) - \text{Arcsin}(z_2) \): we have merely proved that every equation for \( \text{Arcsin}(z_1) + \text{Arcsin}(z_2) \)

\(^7\)As we will see, there is no formal point. However, when \( z_1 \) and \( z_2 \) are close to zero, and we are taking arcsin rather than Arcsin on the left-hand side, then the corresponding signs on the \( \pm \)s agree. In other words, this may give a human a clue.
(4.6.14) \( \text{Arcsinh}(z) = -i \text{Arccos}(iz) \)
(4.6.15) \( \text{Arccosh}(z) = \pm i \text{Arccos}(z) \)
(4.6.16) \( \text{Arctanh}(z) = -i \text{Arctan}(iz) \)
(4.6.17) \( \text{Arccsch}(z) = i \text{Arccsc}(iz) \)
(4.6.18) \( \text{Arcsech}(z) = \pm i \text{Arcsec}(z) \)
(4.6.19) \( \text{Arccoth}(z) = i \text{Arccot}(iz) \)

In terms of another Arcsin must also be an equation for \( \text{Arcsin}(z_1) - \text{Arcsin}(z_2) \). In other words, there is no hope of disentangling the \( \pm \) in equation (28): it must be read as:

\[
\text{Arcsin}(z_1) \pm \text{Arcsin}(z_2) = \text{Arcsin} \left( z_1 \sqrt{1 - z_2^2} + z_2 \sqrt{1 - z_1^2} \right),
\]

with each side taking on eight values modulo \( 2\pi \) (counting cases like \( \text{Arcsin}(1) \) as a “double root”). A similar argument shows that any expression of the form \( \text{Arcsin}(z_1) + \text{Arcsin}(z_2) \subset \text{Arctan}(A) \) must also have \( \text{Arcsin}(z_1) - \text{Arcsin}(z_2) \subset \text{Arctan}(A) \). It is an open question whether there is a “useful” characterisation of \( \text{Arcsin}(z_1) + \text{Arcsin}(z_2) \): of course we could always just convert into Ln form.

For the hybrid equation [2, (4.4.35)] —

\[
\text{Arcsin } z_1 \pm \text{Arccos } z_2 = \text{Arccos} \left( z_1 z_2 \pm \sqrt{(1 - z_2^2)(1 - z_1^2)} \right)
\]

the convention is as in (4.4.32), i.e. the equation cannot be split and \( w^{\frac{1}{2}} \) means \( \sqrt{w} \).

9 Inverse hyperbolic functions

Here there is little new to discuss. There is a very straight-forward\(^8\) relationship between the multi-valued inverse hyperbolic and inverse trigonometric functions, as set out in [2, 4.6.14–19]: The reader may be surprised to see the \( \pm \) in 4.6.14 and 4.6.18, but recall equation (29).

Therefore all that has been said about addition rules, logarithmic representation etc. above, continues to apply.

References


\(^8\)Much more so than in the case of the one-valued functions, where the equivalents of 4.6.15 and 4.6.18 only hold on half the complex plane [5]. In the terminology of [5], all the multivalued function pairs arcsin/arcsinh etc. are count.
A Theorems on inverse trigonometric function

Theorem 1 \cite[4.4.27]{Abramowitz1964}, in the form

\[ \text{Arccos}(z) = -i \ln \left( x + i \sqrt{1 - x^2} \right), \]

(33)

is valid throughout \( \mathbb{C} \).

As before, there is a trivial argument for \( x \) real with \( x^2 \leq 1 \): changing the sign of the Sqrt changes the sign of the imaginary part of the argument of \( \ln \), therefore
changes the sign of the imaginary part of the value of $\text{Ln}$, and therefore the sign of the real part of the answer.

Again, $-\arccos(z) = \ln(x - i\sqrt{1 - x^2})$ is true “up to a constant” on $\mathbb{C}$ by calculus, so it remains to determine this constant in each region. The branch cuts of $\arccos(z)$ are $(-\infty, -1)$ and $(1, \infty)$, which also happen to be the branch cuts of $\sqrt{1 - z^2}$. Call these two lines $L_1$ and $L_2$ respectively. We must also consider the branch cut of $\ln(z - i\sqrt{1 - z^2})$ itself. Now the branch cut of $\ln$ is on the negative real axis, so we want to find those $z$ for which $z - i\sqrt{1 - z^2} = t \in (-\infty, 0)$. This equates to $-i\sqrt{1 - z^2} = t - z$. Squaring both sides, we get a necessary condition of $-1 + z^2 = t^2 - 2tz + z^2$, which reduces to $t^2 - 2tz + 1 = 0$. Hence $z$ has to be totally real. This can be solved to $t = z \pm \sqrt{z^2 - 1}$, which is on the negative real axis if $x \leq -1$. Back-substituting, we find that this is in fact true, though we have to take the $\pm$ as $+$, which means that not much of the branch cut for $\ln$ is actually covered. Call this line $L_3$. Note that it is $(-\infty, -1]$, unlike $L_1$. The union of these branch cuts does not disconnect $\mathbb{C}$, so we have one two-dimensional region and several smaller-dimensional ones.

$\mathbb{C} \setminus (L_1 \cup L_2 \cup L_3)$ Since we have already proved equation (33) for $(-1, 1)$, this extends throughout this region.

$L_1$ Here an evaluation at $z = -2$ is in order. $\arccos(-2)$ is, by the defining equation,

$$\arccos(z) = -i \ln(z + i\sqrt{1 - z^2})$$

(34)

$-i \ln(-2 + i\sqrt{1 - (-2)^2}) = -i \ln(-2 + i\sqrt{3}) = -i \ln(2 + \sqrt{3})$.

Hence the “other value” is $-\pi + i \ln(2 + \sqrt{3})$. $-i \ln(z - i\sqrt{1 - z^2})$ evaluates to $-i \ln(-2 - i\sqrt{3}) = -i \ln(-2 + \sqrt{3}) = \pi - i \ln(2 - \sqrt{3})$. Since $-\ln(2 - \sqrt{3}) = \ln(\sqrt{2} + \sqrt{3})$, this is merely $2\pi$ different from the desired value, which is acceptable.

$\{-1\} = L_3 \setminus L_1$ $\arccos(-1) = \pi$, so negating it gives $-\pi$, which is the same modulo $2\pi$, as we are working. The argument of $\text{Sqrt}$ is zero, so the choice of sign is irrelevant. Hence both sides of equation (33) give a double root.

$L_2$ Here an evaluation at $z = 2$ is in order. $\arccos(2) = -i \ln(2 + i\sqrt{1 - 2^2}) = -i \ln(2 + i\sqrt{3}) = -i \ln(2 - \sqrt{3}) \approx 1.317 \ldots i$. Hence the “other value” is $i \ln(2 - \sqrt{3})$. $-i \ln(z - i\sqrt{1 - z^2})$ evaluates to $-i \ln(-2 - i\sqrt{3}) = -i \ln(2 + \sqrt{3}) = i \ln(2 - \sqrt{3})$. Hence equation (33) is justified here too.

**Theorem 2** [2, (4.4.29)], in the form

$$\text{Arcsec}(z) = -i \ln \left( \frac{\text{Sqrt}(z^2 - 1) + i}{z} \right),$$

(35)

is valid throughout $\mathbb{C}$.

If $z$ is real with $|z| > 1$, then there is the usual easy argument. Changing the sign of the $\text{Sqrt}$ changes the sign of the real part of the input to $\text{Ln}$. Changing
the sign of the whole would add/subtract \( i\pi \) to the value of the Ln, and therefore subtract/add \( \pi \) to the entire right-hand side. So we need to change back the sign of the imaginary part of the input to Ln, i.e. complex conjugate it, which conjugates the value of Ln, i.e. changes the sign of the real part. This is indeed the phenomenon required.

So what we have to prove is that

\[
\pi - \arccsc(z) = -i \ln \left( \frac{-\sqrt{z^2 - 1} + i}{z} \right)
\]

throughout \( \mathbb{C} \). Again this is “true up to a constant” by calculus, so we need to analyse the regions. The branch cut of \( \arccsc \) is \([-1, 1]\), which is also the branch cut of \( \sqrt{z^2 - 1} \). Call this \( L_1 \). We also need to consider the branch cut of the \( \ln \) term. We need to solve

\[
-\sqrt{z^2 - 1} + i = t \in (-\infty, 0).
\]

This is equivalent to \( tz - i = -\sqrt{z^2 - 1} \), and therefore a necessary condition can be obtained by squaring both sides, to get \( t^2z^2 - 2it - z = 0 \). Substituting \( z = x + iy \), the real part is \( t^2x - x \), so \( x = 0 \) (in theory unless \( t = -1 \), but this becomes inconsistent later). The imaginary part is then \( t^2y - 2t - y \), whose solution is \( t = \frac{1 \pm \sqrt{1 + 4y^2}}{2y} \). Since we want \( t \in (-\infty, 0) \), this means that for positive \( y \), we choose \( \pm \) to be \(-\), and vice versa. Back-substituting into equation (36), however, shows that only the case of positive \( y \) works (recall that the conditions were necessary, but not sufficient). Hence the relevant branch cut is the positive imaginary axis — call it \( L_2 \).

We should note that \( 0 \in L_1 \cap L_2 \), but in fact the whole equation is undefined at \( z = 0 \). This therefore effectively splits \( L_1 \) into two parts: call them \( L_1^- \) and \( L_1^+ \). These cuts do not split the complex plane, so there is one two-dimensional region and three one-dimensional regions.

\( \mathbb{C} \setminus (L_1 \cup L_2) \) We have already proved equation (35) on \((-\infty, -1) \cup (1, \infty)\), hence it is valid throughout this region.

\( L_1^- \) Here an obvious evaluation would be at \( z = -0.5 \). \( \arccsc(-0.5) \) is, by the defining equation

\[
\arccsc(z) = -i \ln \left( \frac{\sqrt{z^2 - 1} + i}{z} \right),
\]

\[
-i \ln(-2(\sqrt{-1} + i)) = -i \ln(-\sqrt{-3} - 2i) = -i(\ln(2 + \sqrt{3})) = -\frac{\pi}{2} - i \ln(2 + \sqrt{3}).
\]

Hence the “other value” is \( \pi - \arccsc(0.5) = \frac{3\pi}{2} + i \ln(2 + \sqrt{3}) \).

We have to prove this equal to the evaluation of \( -i \ln \left( \frac{-\sqrt{x^2 - 1} + i}{x} \right) \), viz.

\[
-i \ln(-2(\sqrt{-1} + i)) = -i \ln(-\sqrt{-3} - 2i) = -i(\ln(2 + \sqrt{3})) = \frac{3\pi}{2} + i \ln(2 + \sqrt{3}).
\]

\[18\]
\[-\frac{\pi}{2} - i \ln(2 - \sqrt{3}).\] Since \(\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3}),\) this is equal, modulo \(2\pi,\) to the desired result.

\(L_1^+\) Here an obvious evaluation would be at \(z = 0.5.\) \(\text{arccsc}(0.5) = -i \ln(2(\sqrt{\frac{3}{4}} + i)) = -i \ln(\sqrt{3} + 2i) = -i(\frac{5\pi}{6} + \ln(2 + \sqrt{3})) = \frac{5\pi}{6} - i \ln(2 + \sqrt{3}).\) Hence the “other value” is \(\pi - \text{arccsc}(0.5) = \frac{\pi}{6} + i \ln(2 + \sqrt{3}).\) We have to prove this equal to the evaluation of \(-i \ln\left(\frac{-\sqrt{2} - 1}{z} + 1\right),\) viz. \(-i \ln(-\sqrt{-3} + i) = -i(\frac{5\pi}{6} + \ln(2 - \sqrt{3})) = \frac{\pi}{2} - i \ln(2 - \sqrt{3}).\) Since \(\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3}),\) this is equal to the desired result.

\(L_2\) Here an obvious evaluation would be at \(z = i.\) \(\text{arccsc}(i) = -i \ln\left(\frac{-\sqrt{2} + i}{i}\right) = -i \ln(1 + \sqrt{2}).\) Hence the “other value” is \(\pi - \text{arccsc}(i) = \pi + i \ln(1 + \sqrt{2}).\) We have to prove this equal to the evaluation of \(-i \ln\left(\frac{-\sqrt{2} + i}{i}\right),\) viz.

\[-i \ln\left(\frac{-\sqrt{2} + i}{i}\right) = -i \ln(1 - \sqrt{2}) = -i(i\pi + \ln(\sqrt{2} - 1)) = \pi - i \ln(\sqrt{2} - 1).\]

Since \(\ln(\sqrt{2} - 1) = -\ln(\sqrt{2} + 1),\) the result is proved.

**Theorem 3** [2, (4.4.30)], in the form

\[
\text{Arcsec}(z) = -i \ln\left(\frac{i \text{Sqrt}(z^2 - 1) + 1}{z}\right),
\]

is valid throughout \(\mathbb{C}.\)

If \(z\) is real with \(|z| > 1,\) then there is the usual easy argument. Changing the sign of the Sqrt changes the sign of the imaginary part of the input to Ln, i.e. complex conjugating it, which conjugates the value of Ln, i.e. changes the sign of the real part. This is indeed the phenomenon required.

So what we have to prove is that

\[-\text{arccsc}(z) = -i \ln\left(\frac{-i\sqrt{z^2 - 1} + 1}{z}\right),\]

throughout \(\mathbb{C}.\) Again this is “true up to a constant” by calculus, so we need to analyse the regions. The branch cut of arcsec is \([-1, 1],\) which is also the branch cut of \(\sqrt{z^2 - 1}.\) Call this \(L_1.\) We also need to consider the branch cut of the ln term. We need to solve

\[
\frac{-i\sqrt{z^2 - 1} + 1}{z} = t \in (-\infty, 0).
\]

This is equivalent to \(tz - 1 = -i\sqrt{z^2 - 1},\) and therefore a necessary condition can be obtained by squaring both sides, to get \(t^2z^2 - 2tz + 1 = -(z^2 - 1),\) which simplifies to \(t^2z - 2t + z = 0.\) Substituting \(z = x + iy,\) we see that the imaginary part is \(t^2y + y,\) so \(y = 0 (t^2 + 1 = 0\) is impossible). Solving for \(t\) in terms of \(x,\)
we get \( t = \frac{1 + \sqrt{1 - x^2}}{2} \). Hence \( x \in (-1, 0) \). Call this \( L_2 \). Note that \( L_2 \subset L_1 \), but this does not mean that we can ignore it, since there may be different behaviour on \( L_2 \) and \( L_1 \setminus L_2 \). Note also that, as in the previous theorem, \( z = 0 \) is a point of undefinedness for the whole equation. The branch cuts do not decompose \( \mathbb{C} \).

\( \mathbb{C} \setminus L_1 \) We have already proved the result on \((\infty, -1) \cup (1, \infty)\), so it is true throughout \( \mathbb{C} \).

\( L_2 \) An obvious evaluation point is \( z = -0.5 \). \( \text{arcsec}(-0.5) \) is, by the defining equation

\[
\text{arcsec}(z) = -i \ln \left( \frac{iz^2 - 1 + 1}{z} \right),
\]

where \( -i \ln(-2(i\sqrt{-3/4} + 1)) = -i \ln(-2 + \sqrt{3}) = -i(i\pi + \ln(2 - \sqrt{3})) = \pi - i \ln(2 - \sqrt{3}). \) Hence we have to prove that \( -\pi + i \ln(2 - \sqrt{3}) \) is equal to the evaluation of \( -i \ln \left( \frac{-i\sqrt{z^2 - 1} + 1}{z} \right) \), viz. \( -i \ln(-2(\sqrt{3/4} + 1)) = -i \ln(-2 + \sqrt{3}) = -i(i\pi + \ln(2 + \sqrt{3})) = \pi - i \ln(2 + \sqrt{3}) = \pi - i \ln(2 - \sqrt{3}). \)

Since we are working modulo \( 2\pi \), this proves the desired result.

\( L_1 \setminus L_2 \) An obvious evaluation point is \( z = 0.5 \). \( \text{arcsec}(0.5) \) is \( -i \ln(2(i\sqrt{-3/4} + 1)) = -i \ln(2 - \sqrt{3}) \). Hence we have to prove that \( i \ln(2 - \sqrt{3}) \) is equal to the evaluation of \( -i \ln \left( \frac{-i\sqrt{z^2 - 1} + 1}{z} \right) \), viz. \( -i \ln(2(-i\sqrt{-3/4} + 1)) = -i \ln(2 + \sqrt{3}) = i \ln(2 - \sqrt{3}) \) as required.