Why is quantifier elimination doubly exponential?

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Notation

- d The maximum degree (in each variable separately) of the input polynomials. $\mathfrak{d} \leq dn$ total degree.
- l The maximum bit-length of the integer coefficients
- m The number of (distinct) polynomials.
- n The number of variables.
- a The number of alternations of quantifiers. $a \leq n - 1$.
- c The number of equational constraints.
- (M, D) At most M sets, each of combined degree $\leq D$ [McC84].

This is the standard theory setting. Real problems tend to involve rational functions, and rational, or even algebraic, numbers. See [UDE22].

The complexity of QE (and hence CAD) is doubly exponential in n, more precisely $d^{2^{e_d}} m^{2^{e_m}}$ where e_d and e_m depend non-trivially on n (or on a). What are e_d, e_m?

Quantifier Elimination

Given a quantified statement in $n = k + l$ variables

$$Q_1 x_1 \cdots Q_k \Phi(x_1, \ldots, x_k, y_1, \ldots, y_l), \qquad Q_i \in \{\exists, \forall\}$$

find an equivalent quantifier-free formula $\Psi(y_1, \ldots, y_l)$. Our applications will be either **C** (with $+, -, \times, =, \neq$) or **R** (with $+, -, \times, =, \neq, >, \geq, <, \leq$).
Note the absence of division: philosophical and practical issues here [UDE22].
R implies **C** (take real and imaginary parts, and you get $|z|$ and $\overline{z}$ for free). **C** with $\overline{z}$ implies **R**.
Solved by [Tar51, Sei54], but indescribable complexity.
First plausible solution [Col75], via cylindrical algebraic decomposition (CAD), constructed via projection/lifting.

CAD Sign invariant for P

Decomposition: $\mathbf{R}^n = \bigcup_i C_i$ and $i \neq j \Rightarrow C_i \cap C_j = \emptyset$

Cylindrical: If P_m is the projection onto the first m variables, then either $P_m(C_i) = P_m(C_j)$ or $P_m(C_i) \cap P_m(C_j) = \emptyset$.

Also A sample point in each cell, arranged cylindrically.

In fact This is slightly stronger than we need: can relax to "block-cylindrical", where m has to be where $Q_m \neq Q_{m+1}$, i.e. the quantifiers alternate.

(Semi-)Algebraic The boundaries of each cell are semi-algebraic functions, i.e. defined by polynomials and $=, \neq, >, \geq, <, \leq$.

N.B. This is the standard definition, but permits many pathological examples that "no sane algorithm would construct". See [DLS20].

Sign invariant In each C_i every $P_j \in \mathcal{P}$ is positive, or negative, or identically zero, so sample points suffice, and $\forall x_i$ translates to "$\forall x_i$ i-th coordinates of sample points".

Projection/Lifting for a property Z

Given $\mathcal{P}_v$ polynomials in v variables, construct a set $\mathrm{Proj}(P_v)$ in $v-1$ variables such that a CAD of $\mathbf{R}^{v-1}$ Z-invariant for $\mathrm{Proj}(P_v)$ can be lifted to a CAD of $\mathbf{R}^v$ Z-invariant for P_v.

[Col75] Z is "sign". Because a polynomial might vanish identically on a cell, also take subresultants, so $e_m \approx n \log_2 3$.

[McC85] Z is "order". Might fail if a polynomial vanishes identically on a cell. But $e_m \approx n$.

[Bro01] Z is "order", but projection is cheaper.

[Laz94, MPP19] Z is "lex-least", and (Lazard lifting) if a polynomial vanishes identically, divide out the obstruction. Again $e_m \approx n$.

[BM20] Z is "lex-least", but projection is cheaper.

Proj always involves $\mathrm{disc}_{x_v}(p_i)$ and $\mathrm{res}_{x_v}(p_i, p_j)$, hence both degree and number of polynomials squares with each projection.

The problem with iterated resultants

Consider $f_1, f_2, f_3 \in \mathbf{Q}[x, y, z]$ of degree d in each variable.
Then $\mathrm{res}_z(f_1, f_2)$ etc. have degree $2d^2$, and
$R := \mathrm{res}_y(\mathrm{res}_z(f_1, f_2), \mathrm{res}_z(f_1, f_3))$ has degree $8d^4$.
[And so $e_d \approx n$]
But (Bézout) $f_1 = f_2 = f_3 = 0$ has $\le 27d^3$ points (x, y, z).
The problem is that R has as roots

$$\begin{aligned}
\text{(true)}\quad & x : \exists y \exists z f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z) = 0 \\
\text{(spurious)}\quad & x : \exists y\, [\exists z_1 f_1(x, y, z_1) = f_2(x, y, z_1) = 0] \wedge \\
& \qquad [\exists z_2 f_1(x, y, z_2) = f_3(x, y, z_2) = 0].
\end{aligned}$$

In this case, a Gröbner base [EBD20], or even
gcd $(R, \mathrm{res}_y(\mathrm{res}_z(f_1, f_2), \mathrm{res}_z(f_2, f_3)))$, will solve the problem. Goes some way to explain [McC99a]'s observation that iterated resultants tend to factor.

But in the general case, those "spurious" roots are where the projected topology of $V(f_i)$ changes.

Equational Constraints

[Col98] What if our formula Φ is $f = 0 \wedge \hat{\Phi}$, where $\hat{\Phi}$ involves $m-1$ polynomials g_i?

[McC99b] Answers this: we only need $O(m)$ $\mathrm{res}_x(f, g_i)$, not $O(m^2)$ $\mathrm{res}_x(g_i, g_j)$, since

$$\mathrm{res}_x(g_i, g_j)|_{f=0} \propto \mathrm{res}_y(\mathrm{res}_x(f, g_i), \mathrm{res}_x(f, g_j)). \qquad (1)$$

Means that, after the x projection, we only have $O(m)$ polynomials not $O(m^2)$.

[McC01] Generalises to $f_1 = 0 \wedge \cdots \wedge f_c = 0 \wedge \hat{\Phi}$.

$+$ Reduces e_m from n to $n-c$, nothing for e_d.

[BDE+16] Generalises to where only part of the formula has equational constraints: "truth-table invariant CAD"

[EBD20] Can use Gröbner bases, rather than just iterated resultants, to reduce degree growth, *ideally* e_d becomes $n-c$.

But All this is for the McCallum projection, i.e. well-oriented.

Doesn't Lazard projection/lifting eliminate "well-oriented"?

- $+$ Yes, for straight cylindrical algebraic decomposition
- **But** if $f(x, y, z, \ldots)$ vanishes identically on some surface $S(y, z, \ldots)$, the constant of proportionality in (1) is 0, and we learn nothing about $\mathrm{res}_x(g_i, g_j)$ from $\mathrm{res}_x(f, x_i)$.
- ⚠ "Nullification" has come back to bite us, but only nullification of f, not the g_i.
- **Call** S the *foot* of the "curtain": the "vertical" part of $f = 0$ [NDS20].
- **$\dim(S)$** The case $\dim(S) = 0$ is tractable [Nai21] — see that thesis for more details of $\dim(S) > 0$.

Graph Theory to the rescue?

Instead of considering degrees of the polynomials in F, consider the graph $\mathcal{G}(F)$ on $\{x_1, \ldots, x_n\}$ with an edge betwen (x_i, x_j) iff there is a polynomial in F contaning both x_i and x_j.
Connectedness?

Gröbner
: If $\mathcal{G}(F)$ is not connected, the problems are independent, and [Buc79, Criterion 1] will treat them as such.

CAD
: Essentially independent, but this is hard to describe: we have "the outer product" of the two (or more) CADs. We definitely need to project one component at a time.

Problem

Recognise, and treat effectively, this case, also "nearly disconnected" (see next)

Graph Theory to the rescue continued

A graph $\mathcal{G}$ is *chordal* if every >3-cycle has a chord. Equivalently, every induced cycle has length 3. Every graph $\mathcal{G}$ has a chordal completion $\overline{\mathcal{G}}$.

Minimum chordal completion is NP-complete [Yan81], but that doesn't really worry me: minimal will probably do.

If this is the complete graph, then graph theory doesn't seem to help us: the exciting case is when $\overline{\mathcal{G}}$ is smaller.

An ordering $\succ$ on the vertices $x_1, \ldots, x_n$ is a *perfect elimination ordering* (PEO) if $\forall i$ x_i and its neighbours $x_j : x_j \prec x_i$ form a clique. This, and chordality, can be found efficiently [RTL76].

Let n' be the maximal length of a path from x_1 to x_n (as reordered) in $\mathcal{G}$ following $\succ$.

Graph Theory to the rescue continued

Non-trivial chordality has been exploited.

Regular Chains [Che20] shows how it can be exploited efficiently.

Gröbner Bases [CP16] consider "chordal elimination". The challenge here is that an S-polynomial can introduce new edges in $\mathcal{G}$.

Triangular Chordality preserving is proved in [MBL21].

CAD [LXZZ21] consider chordality, ordering x_i in a perfect elimination ordering, then essentially use the same algorithm.

e_d is now n' rather than n (polynomials "drop through" layers!).

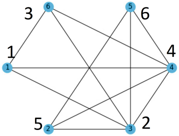
The quantifier structure may be incompatible with the perfect elimination ordering.

What we currently lack is any view of how common in practice these non-trivial chordal structures are, but they are related to "nearly disconnected" $\mathcal{G}$.

But [DM22] in CASC 2022 (being digested)

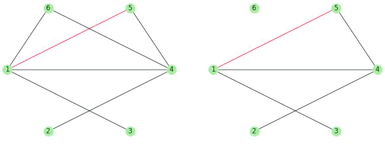
- [MBL21] proves that (sparse) triangular decomposition following a PEO preserves chordal structure.
- But when run in practice, they observe new edges.

Original Chordal Graph



?swap 2,3

Graphs of triangular decompositions



(a) (b)

Extra lines in red

There are four issues:

- Simplifying a Polynomial Set with Its Binomials;
- Simplifying a Polynomial System with Binomials;
- Reducing Inequation Polynomials with a Polynomial in the TS;
- Reducing a TS with a Polynomial in the TS.

? But is it safe to do these as a post-process?

CGB=Comprehensive Gröbner Bases (I) [Wei98, FIS15]

The key idea is this. We consider an "innermost block" in this form:

$$\exists \overline{x} \left(\begin{array}{c} f_1(\overline{y},\overline{x}) = 0 \wedge \cdots f_r(\overline{y},\overline{x}) = 0 \wedge \\ p_1(\overline{y},\overline{x}) > 0 \wedge \cdots p_s(\overline{y},\overline{x}) > 0 \wedge \\ q_1(\overline{y},\overline{x}) \neq 0 \wedge \cdots q_t(\overline{y},\overline{x}) \neq 0 \end{array} \right)$$

where $\overline{y}$ represents the remaining variables, and $f_i, p_j, q_k \in \mathbf{Q}[\overline{y},\overline{x}] \setminus \mathbf{Q}[\overline{y}]$. We introduce new variables $\overline{z}$ and $\overline{w}$, with $\overline{z},\overline{w} \succ \overline{x}$, and consider the polynomials

$$\{f_1,\ldots,f_r,\underbrace{z_1^2p_1-1,\ldots,z_s^2p_s-1}_{\text{forcing positive}},\underbrace{w_1q_1-1,\ldots,w_tq_t-1}_{\text{forcing nonzero}}\}.$$

Let $\mathcal{G} = (S_i, G_i)$ be a Comprehensive Gröbner System (with parameters $\overline{y}$) for this so that $\overline{y}$ space is partitioned by the S_i. We claim each G_i will be $\{f'_1,\ldots,f'_{r'},u_1z_1^2-p'_1,\ldots,u_sz_s^2-p'_s,v_1w_1-q'_1,\ldots,v_tw_t-q'_t\}$. Our answer will be $\bigvee_i \Psi_i(S_i,G_i)$: next two slides explain Ψ_i.

G_i zero-dimensional ($\overline{z}, \overline{w}$ irrelevant for dimension)

If $G_i = (1)$ then we return `false`. Otherwise recall
$G_i = \{f'_1, \ldots, f'_{r'}, u_1 z_1^2 - p'_1, \ldots, u_s z_s^2 - p'_s, v_1 w_1 - q'_1, \ldots, v_t w_t - q'_t\}$.
Let $I = \langle f'_1, \ldots, f'_{r'} \rangle$,

$$\chi(x) = \prod_{(e_1,\ldots,e_s)\in\{0,1\}^s} \chi^I_{(p'_1/u_1)^{e_1},\cdots,(p'_s/u_s)^{e_s}}(x) = x^{2^s d} + \sum_0^{2^s d-1} a_i x^i.$$

The answer is $\Psi_i := \mathcal{F}(S_i) \wedge I_{2^s d}(a_i)$.
JHD: at least that's my reconstruction. I can't see where the w_i (the $\neq 0$) terms come in. Also, the subscript of $\chi^I_{\ldots}$, the characteristic polynomial of $M^I_{\ldots}$, is not a polynomial.

$\exists\phi$: $G_i > 0$-dimensional ($\overline{z}, \overline{w}$ irrelevant for dimension)

$\overline{u} :=$ maximal independent variables $(\overline{x}, G_i, \succ)$. (B)
If $\overline{u} = \overline{x}$ return SYNRAC$(\mathcal{F}(S) \wedge \exists\overline{x}\phi)$ [Wei98]
$\overline{x}' := \overline{x} \setminus \overline{u}$; $\phi_1 := \texttt{Free}(\phi, \overline{x}')$; $\phi_2 := \texttt{NonFree}(\phi, \overline{x}')$;
$\varphi := \phi_1 \wedge \text{Recurse}(S_i, \exists\overline{x}'\phi_2)$ (1)(A)
JHD: I think this means φ now only contains $\overline{u}$-variables
Let $\varphi_1 \vee \cdots \vee \varphi_l$ be a disjunctive normal form of φ. (C)
for $1 \le j \le l$ **do**
$\quad\varphi_j^{(1)} := \texttt{Free}(\varphi, \overline{u})$; $\varphi_j^{(2)} := \texttt{NonFree}(\varphi_j, \overline{u})$;
$\quad\psi_j := \varphi_j^{(1)} \wedge \text{Recurse}(S_i, \exists\overline{u}\phi_j^{(2)})$ (2)(E)
Return $\Psi := \mathcal{F}(S_i) \wedge (\psi_1 \vee \cdots \vee \psi_l)$
JHD: "Recurse" goes right back to the MainQE, note that call (1) has pushed the $\overline{u}$-variables into being parameters (I think) (D). But somehow S_i gets lost in these recursions: I hope I've added it in the right place. Their Theorem 16 states that this does terminate — far from obvious (F).

CGB=Comprehensive Gröbner Bases (IV) [Wei98, FIS15]

- A. Recursing with S is, I think, my interpolation to make sense of the recursions we'll see later. S initially is $\mathbf{R}^{\#\overline{y}}$.
- B. There's a lot of freedom here: ML?
- C. Note that our main recursion is on ϕ in conjunctive normmal form (CNF), whereas here we convert to disjunctive normal form (DNF) and implicitly back at the end of the block. Since CNF$\leftrightarrow$DNF naïvely is exponential, this would provide an exponential blowup at each $\exists/\forall$ boundary, similar to [DH88].
- D. Therefore this recursion is on strictly fewer variables, since dim > 0.
- E. Therefore this recursion is on strictly fewer variables, since $\overline{u} \neq \overline{x}$. $\varphi_j^{(1)}$ is free of $\overline{u}$ by construction, and free of $\overline{x}'$ since it comes from ϕ_1, so actually belongs in an outer block. We might ask why such things exist, but they could be generated by the recursion.
- F. But the two previous notes are probably key.

Complexity of CGB

I know no results on the complexity of Comprehensive Gröbner Bases/Systems.
Since we are doing Gröbner Bases, we might *hope for* singly exponential behaviour at each block, and hence $e_d = O(a)$ rather than $O(n)$, but worst-case Gröbner bases can be doubly exponential [MR13]. *If* we get $O(a)$ behaviour, though, this does not depend on having a lot of equational constraints.
We are doing CNF/DNF conversions at each quantifier alternation, as with VTS, so this could be expected to give us $e_m = O(a)$ rather than $O(n)$.

it's not $\mathbf{R}/\mathbf{C}$: it's quantifiers (and alternations)

[DH88, BD07] Are really about the combinatorial complexity of quantifier alternations

Let $S_k(x_k, y_k)$ be the statement $x_k = f(y_k)$ and then define recursively $S_{k-1}(x_{k-1}, y_{k-1}) := x_{k-1} = f(f(y_{k-1})) :=$

$$\underbrace{\exists z_k \forall x_k \forall y_k}_{Q_k} \underbrace{\left(\underbrace{(y_{k-1} = y_k \wedge x_k = z_k}_{L_k^1}) \vee (\underbrace{y_k = z_k \wedge x_{k-1} = x_k}_{L_k^2}) \right)}_{L_k} \Rightarrow S_k(x_k, y_k).$$

We can transpose this to the complexes, and get zero-dimensional QE examples in $\mathbf{C}^n$ with $2^{2^{O(n)}}$ isolated point solutions, roots of an irreducible polynomial of that degree [DH88]. Or can get that many even though the equations are all linear and the solution set is zero-dimensional [BD07].

Two iterations of Heintz:

$$\exists z_1 \forall x_1 \forall y_1 \left[\left(L_1^1 \vee L_1^2 \right) \Rightarrow \exists z_2 \forall x_2 \forall y_2 \left[\left(L_2^1 \vee L_2^2 \right) \Rightarrow \Phi \right] \right]$$

which becomes

$$\exists z_1 \forall x_1 \forall y_1 \exists z_2 \forall x_2 \forall y_2 \left(L_1^1 \vee L_1^2 \right) \Rightarrow \left[\left(L_2^1 \vee L_2^2 \right) \Rightarrow \Phi \right].$$

The quantified part is then

$$\left(\neg L_1^1 \wedge \neg L_1^2 \right) \vee \left(\neg L_2^1 \wedge \neg L_2^2 \right) \vee \Phi.$$

We will get singly-exponential blow-up as we convert this to Conjunctive Normal Form

Other questions than QE: multistaionarity

Consider ([BDE+17]) a single semi-algebraic set defined by

$$f_1(x_1,\ldots,x_{n-1},k_1)=0 \land f_2(x_1,\ldots,x_{n-1},k_1)=0\land\cdots$$
$$f_{n-1}(x_1,\ldots,x_{n-1},k_1)=0\land x_1>0\land\cdots\land x_{n-1}>0$$

and ask the question "How does the number of solutions vary with k_1?" The f_i are multilinear ($d=1$ but $\mathfrak{d}=2,3,4$) and primitive, and are pretty "generic".

Of course, this doesn't guarantee that all the iterated resultants in [EBD15], or the Gröbner polynomials in [ED16], are primitive, but in practice they are.

In practice can handle k_1, k_2 and looking at k_1, k_2, k_3. But note we want $k_1,\ldots,k_{19}$ for the real biochemical application.

Questions

1. Can we actually say anything about the complexity of GB methods [EBD20]?
2. What happens when we have equational constraints that don't involve the first projection variable?
3. Can we actually say anything about the complexity of CGB-based methods for QE?
4. Can CGB methods, which do QE, actually produce block-cylindrical algebraic decompositions? If so, this would be the first real construction here.
5. Chordality: understand [DM22].
6. Are there any "weak average case complexity" [AL17] results? The examples of [BD07, DH88] seem very special.
7. Understand "which no sane algorithm would construct".

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