GKS on flatness

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Advertisement for what is well known among commutative algebraists.

Example 1 *Takes variety of three points* $(-1, 1)$, $(1, \pm 1)$*, which is not equiprojectable onto the x-axis: in GKS' language, "the fibres above the two points* 1*,* -1 *are not the same".*

This works in 0-dimensional cases: what is the appropriate generalisation?

The ground field won't really matter. One can think of it as being algebraically closed (GKS' normal mode of thought), or possibly $\mathbf{R}$.

Flatness

We have a map $f : X \to B$ of algebraic varieties (or possibly schemes, if multiplicity matters). What does it mean to say that the fibres of f "vary continuously"?

To get anywhere sensible, I need B to be irreducible (possibly irreducible over the algebraic closure). Therefore in the example I just had two points, which are disconnected, so I don't get anywhere new.

Topologists tend to ask for fibre bundles, i.e. locally trivial, and this is too strong for us.

Example 2 *Let* X *be* $(x, y, z)M \begin{pmatrix} x \\ t \\ z \end{pmatrix} \subset \mathbf{P}^2$*, and we're varying* M*. and consider* $f : X \to \det(M)$*. More simply* $\{(x, y, t) : x^2 + y^2 + t = 0\} \subset \mathbf{A}^2$*. Conic, or two lines when* $t = 0$*? Over the complexes this is good, even though the conic might degenerate at one point. How much does this worry you?*

Example 3 *Elliptic curve* $y^2 z = 4x^3 + g_2 x z^2 + g_3 z^3$*. The invariant is* $j : (g_2^3 : g_2^3 - 27g_3^2) \in \mathbf{P}^1$ *(up to a factor of 1728, which makes one worry about characteristics 2,3 — not germane, though). Normally elliptic curve, but special when* $j = \infty$ *(*$j = 0$ *is curious*[1]*, but a smooth curve). Again, how much does this worry you?*

[1] The one with an automorphism of order 3.

Unexample 1 $\{xy = 0\}$, *which is a point and a line. Note that* X *in this case is reducible.*

JHD thinks this is definitely bad, as there is a change in dimension: generically zero, but one above $x = 0$.

GKS My condition is close to equi-dimensional, but is well-studied.

Unexample 2 *Blow-up:* $\mathbf{A}^2 \times \mathbf{P}^1 = (x_1, x_2) \times (y_1, y_2) \supset X = \{x_i y_j = x_j y_i\}$. *The general fibre is a point, and the special fibre is a whole* $\mathbf{P}^1$*. This is an irreducible variety, unlike the previous case..*

Algebraic idea

Definition 1 *Let* A *be a commutative ring, and an* A*-module* M*. We say that* M *is flat over* A *if* $M \otimes_A$ *is exact. This means in practice that if* $N_1 \hookrightarrow N_2$*, then* $M \otimes_A N_1 \hookrightarrow M \otimes_A N_2$ *(*$\hookrightarrow$ *signifies injectivity).*

For example, if V in $\mathbf{R}^n$, then $V \otimes_{\mathbf{R}} \mathbf{C}$ is the "corresponding $\mathbf{C}$-vector space": essentially a change of coefficients. "Reduction mdulo p" is tensoring a $\mathbf{Z}$-module with '$\mathbf{Z} \pmod 5$'.

It is enough to check that for a an ideal of A, $a \otimes_A M \to M$ is injective. This means that we don't need to quantify over all N_1, only ideals of A, which is much more computable. There must be code in Macauley or Magma which will do this.

What's so good about flatness? See [Har77]—algebraic geometer's bible. Let$f : X \to B$ be a morphism of schemes (possibly non-reduced, though I think I'd like B to be irreducible).

Definition 2 f *is flat over* B *at some* $x \in X$ *if the local ring of* X *at* x*,* $O_{X,x}$ *is flat as an* $O_{B,f(x)}$*-module.*

This is a nice algebraic definition, but not very geometric.

Definition 3 f *is flat if it is flat at every* $x \in X$*.*

For schemes of finite type (a noetherian-like condition) and irreducible Y, we have that the following are equivalent

(i)–global Every irreducible component of X has dimension $\dim B + n$ [trivially true if X is irreducible]

(ii)–local At every point $t \in B$, every irreducible component of $f^{-1}(t)$ has dimension n.

Note that Unexample 2 is an example of non-flatness. Note, however, that, even if X is irreducible, the fibres might not be irreducible, but they'll still have the "right" dimension: see Example 2. The backwards direction tells you that the local picture "reassembles" correctly.

Theorem 1 *Let B be an integral Noetherian scheme. Let $X \subset \mathbf{P}^n_B$ be a closed sub-scheme. The for each $t \in B$, consider the Hilbert polynomial $P_t(z)$ of $X(t) \subset \mathbf{P}^n_{k(t)}$. Then X is flat over B iff P_t is independent of t.*

T is the base, and $\mathbf{P}^n_B$ is the universal projective space over B. The Hilbert polynomials tell you "what happens in the long term". NB only true projectively, and we're looking at Hilbert polynomials,not functions. Note that, in Example 1, the Hilbert function is a constant, either 1 or 2.

JHD It seems clear that, even if flatness is not what we want, we should be able to say *why* it's not. It's the elephant in the room.

GKS It's an algebraic condition, but with geometric consequences.

JHD This is exactly the sort of thing we want.

JHD (afterthought) A common case for us is that f is projection onto the first d coordinates, and B is K^d (and therefore irreducible).

References

[Har77] R. Hartshorne. Algebraic Geometry. *Springer-Verlag*, 1977.