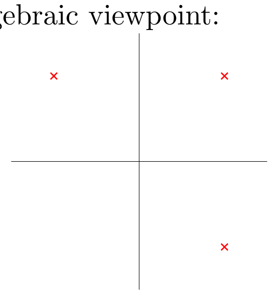
Triangular Sets Seminar

Seminar 4: James H. Davenport - 24/10/2011

Let us head back to the “three point” example and look at it from an algebraic viewpoint:



How do we define it?

Gröbner Bases

We can define it as an ideal:

$$\langle F\rangle = \langle x^2-1, (x+1)(y^2-1)+(x-1)(y-1)\rangle$$

and we can put it into a Gröbner Basis algorithm (in this case it doesn't matter which ordering we take) and get

$$G = \{x^2-1, (x-1)(y-1), y^2-1\}.$$

F is a triangular set and what we would like to do is solve for x, then back-substitute (much like a system of linear equations). However, the degree in y of the second equation depends on which x we choose. This is dealt with in the Gianni-Kalkbrener Theorem (discovered independently in [2] and [3]), which applies when G is a plex Gröbner Basis of a zero-dimensional ideal.

Theorem 1 (Gianni-Kalkbrener Theorem - Gröbner Basis version)**.** *Let our variables be ordered $x_1 > x_2 > \cdots > x_n$ and let our plex Gröbner basis of a zero-dimensional ideal be G. Write G in the following form:*

$$G = \left\{ \begin{array}{l} p_n(x_n) \\ p_{n-1,1}(x_{n-1}, x_n), \ldots, p_{n-1,k_{n-1}}(x_{n-1}, x_n) \\ p_{n-2,1}(x_{n-2}, x_{n-1}, x_n), \ldots, p_{n-2,k_{n-1}}(x_{n-2}, x_{n-1}, x_n) \\ \vdots \end{array} \right\} \begin{array}{l} \}\mathbb{B}_n \\ \}\mathbb{B}_{n-1} \\ \}\mathbb{B}_{n-2} \\ \vdots \end{array}$$

so that each $\mathbb{B}_i = G \cap \mathbb{K}[x_i, \ldots, x_n]$ and we sort the elements of $|mathbbB_i$ by increasing degree in x_i.

Then given any solution $\alpha = (\alpha_{k+1}, \ldots, \alpha_n)$, a solution of $\mathbb{B}_{k+1}$ we define

$$\begin{array}{rcl} \phi_\alpha : \mathbb{K}[x_1, \ldots, x_n] & \longrightarrow & \mathbb{K}[x_1, \ldots, x_n] \\ x_i & \longmapsto & \alpha_i \end{array}$$

Then:

- *For all $p \in \mathbb{B}_k$ we have*

$$\phi_\alpha \left(lc_{x_k}(p) \right) = 0 \qquad \Longleftrightarrow \qquad \phi_\alpha \left(p \right) = 0$$

- *If $p \in \mathbb{B}_k$ is the first polynomial (in our ordering) that does not vanish under ϕ_α, then*

$$\phi_\alpha(p) \mid \phi_\alpha(q) \qquad \forall q \in \mathbb{B}_k$$

We also recall one of the nicest properties of Gröbner Bases (indeed, sometimes it is thought of as part of the definition of a Gröbner Basis):

Theorem 2 (The Hilbert Basis Theorem)**.** *For a Gröbner basis G we have:*

$$\langle lm\,(G) \rangle = \langle lm\,(\langle G \rangle) \rangle$$

In short, Theorems 1 and 2 tells you that there are *no* surprises if you use Gröbner Bases — you cannot be lead down a dead end.

Triangular Sets and Regular Chains

Definition. A set of polynomials is a *triangular set* if it has disjoint main variables (we will order the variables $x_1 > x_2 > \cdots > x_n$). A variable which is the main variable of an element of T we say is *algebraic with respect to* T. We denote the set of algebraic variables of T as $\mathrm{algVar}(T)$. Any variable which is not algebraic we call a *parameter*.

Definition. For a polynomial p with main variable x we define the *initial of* p, denoted $\mathrm{init}(p)$ to be

$$p = \mathrm{init}(p) \cdot x^k + \dots$$

Definition. A triangular set $T = \{t_1, \dots t_k\}$ (where $\mathrm{mvar}(t_1) > \mathrm{mvar}(t_2) > \cdots$) is a *regular chain* if and only if

- $\widehat{T} := \{t_2, \dots, t_k\}$ is a regular chain;
- $\mathrm{init}(t_1)$ is regular with respect to $\widehat{T}$ (i.e. not a zero divisor in $\widehat{T}$)

We have a version of the Gianni-Kalkbrener Theorem in terms of Regular Chains:

Theorem 3 (Gianni-Kalkbrener Theorem - Regular Chains version)**.** *A zero-dimensional variety can be written as the disjoint union of solutions of regular chains.*

This is an easy consquence of Theorem 1 by taking the polynomials defined in the second part of the theorem.

Example 1. For our "three point" example we obtain the decomposition:

$$V(F) = V(x-1, y^2-1) \,\dot{\sqcup}\, V(x+1, y-1)$$

Multiplicity and Positive Dimensions

What about multiplicity? Theorem 1 preserves multiplicity; this is due to the fact a Gröbner basis generates the ideal rather than the radical of the ideal.

What can we say about multiplication in Theorem 3?

Can we extend to positive dimensions? The key paper is [1] which will be covered by JHD next week.

References

[1] Elisabetta Fortuna, Patrizia Gianni, and Barry Trager. ScienceDirect - Journal of Pure and Applied Algebra : Degree reduction under specialization. *Journal of pure and applied algebra*, 2001.

[2] P. Gianni. Properties of Gröbner Bases under specializations. *Proc. EUROCAL 1987*, pages 293–298.

[3] M. Kalkbrener. Solving systems of algebraic equations by using Gröbner bases. *Proc. EUROCAL 1987*, pages 282–292.