OpenMath and MathML

Differentiating between analysis and algebra

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(Thanks to RJB for the improved title)

University of Bath

2 October 2008 (slides updated 4th)

Introduction (1)

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- At the other extreme, we might want a fully formalised system such as COQ or NuPRL.
- In between, we might want what de Bruijn called “the mathematical vernacular”.

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... the deism of Liebniz over the dotage of Newton ...
[Babbage, chapter 4]

A nontrivial issue: differentiation

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How can this be: surely we all know what differentiation is!

$\frac{\mathrm{d}}{\mathrm{d}x}x^2$ — MathML

```
<apply>
  <diff/>
  <bvar><ci>x</ci></bvar>
  <apply>
    <power/>
    <ci>x</ci>
    <cn>2</cn>
  </apply>
</apply>
```

$\frac{\mathrm{d}}{\mathrm{d}x}x^2$ — OpenMath

```
<OMA>
  <OMS cd="calculus1" "name="diff"/>
  <OMBIND>
    <OMS cd=" fns1" name="lambda"/>
    <OMBVAR><OMV name="x"/></OMBVAR>
    <OMA>
      <OMS cd="arith1" name="power"/>
      <OMV name="x"/>
      <OMI>2</OMI>
    </OMA>
  </OMBIND>
</OMA>
```

Only trivial syntactic differences?

```
<apply>
  <diff/>


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Not so! consider $\sin' = \cos$

```
<OMA>
  <OMS cd="relation1" "name="eq"/>
  <OMA>
    <OMS cd="calculus1" "name="diff"/>
    <OMS cd="transc1" name="sin"/>
  </OMA>
  <OMS cd="transc1" name="cos"/>
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whereas MathML can only encode $(\sin x)' = \cos x$.

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i.e. OpenMath's differentiation is a functional.

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- What you learned in calculus, which *I* shall write as $D_{\epsilon\delta}$: the "differentiation of ϵ–δ analysis". Also $\frac{\mathrm{d}}{\mathrm{d}_{\epsilon\delta}x}$, and its inverse ${}_{\epsilon\delta}\int$.
- What is taught in differential algebra, which *I* shall write as D_{DA}: the "differentiation of differential algebra". Also $\frac{\mathrm{d}}{\mathrm{d}_{\mathrm{DA}}x}$, and its inverse ${}_{\mathrm{DA}}\int$.

$D_{\epsilon\delta}$ (for functions $\mathbf{R} \to \mathbf{R}$)

Define $\mathrm{CL}(f, x_0)$ (the "Cauchy Limit") as

$$\mathrm{CL}(f, x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and $D_{\epsilon\delta}(f) = \lambda x.\, \mathrm{CL}(f, x)$.

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- $D_{\epsilon\delta}(\lambda x. f(g(x))) = D_{\epsilon\delta}(g)\lambda x. D_{\epsilon\delta}(f)(g(x))$. (Chain rule)

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Note that there is no Chain Rule as such, since composition is not necessarily a defined concept on R.

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(at least up to removable singularities).

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As currently defined, OpenMath's `diff` is (much closer to) $D_{\epsilon\delta}$, MathML's is (much closer to) D_{DA}, and we actually *do* need both. If you don't believe this, let's look at integration.

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What is naturally defined is integration over an interval I. We let D stand for sub-divisions $d_1 = a < d_2 < \cdots < d_n = b$ of $I = [a, b]$, and $|D|$ for the largest distance between neighbouring points in D, i.e. $\max_i(d_{i+1} - d_i)$.

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Then ${}_{\epsilon\delta}\int_I f = \liminf_{|D| \to 0} \overline{S_D} = \limsup_{|D| \to 0} \underline{S_D}$ *if* both exist and are equal.

Consequences: Fundamental Theorem of Calculus (FTC)

Define ${}_{\epsilon\delta}\int_a^b f = \begin{cases} {}_{\epsilon\delta}\int_{[a,b]} f & a \leq b \\ -{}_{\epsilon\delta}\int_{[b,a]} f & a > b \end{cases}$

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(Of course, this is normally stated without the λ.)

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If g and h are two such integrals, then $D_{\mathrm{DA}}(g - h) = 0$, i.e. "$g$ and h differ by a constant".
One difficulty is that this is really a "$\mathrm{constant}_{\mathrm{DA}}$" (something whose D_{DA} is zero), and, for example, a Heaviside function is a $\mathrm{constant}_{\mathrm{DA}}$, though not a constant in the usual sense.

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Note the caveat on continuity: $g : x \mapsto \arctan\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$ ($\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = \frac{-\pi}{2}$ whereas $\lim_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$), which accounts for the invalidity of deducing that the integral of a negative function is positive —

$$\int_{-1}^{1} \frac{-1}{x^2+1} = \mathcal{I}(g)(1) - \mathcal{I}(g)(-1) = \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2} > 0.$$