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Amdahl's Law reveals a natural upper bound on the speedup that is theoretically possible even before we add in implementation overheads

Suppose we have a problem of which 90% can be run in parallel, leaving 10% sequential code

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We still have the 10% sequential part

So the speedup is

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A speedup of 10 even on an infinite number of processors

It doesn't even matter what the problem is, or what hardware we have

This is Amdahl's Law:

Every program has a natural limit on the maximum speedup it can attain, regardless of the number of processors used

We can quantify Amdahl's Law:

Let $T = T_{seq} + T_{par}$ be the time spent in the sequential and parallel parts of our problem on a sequential processor

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Then the *maximum* speedup S_p using *p* processors on the parallel part is

$$\mathcal{S}_{\mathcal{p}} \leq rac{\mathcal{T}_{\mathsf{seq}} + \mathcal{T}_{\mathsf{par}}}{\mathcal{T}_{\mathsf{seq}} + \mathcal{T}_{\mathsf{par}}/\mathcal{p}}$$

where we have perfectly parallelised the parallel part

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This limit is determined by the time taken in the sequential part of the computation

To see this consider the fraction $x = T_{seq}/(T_{seq} + T_{par})$ which is the proportion of the sequential part within the whole

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And so

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4

is bounded

Note that Amdahl does not say anything about how the speedup varies with \ensuremath{p}

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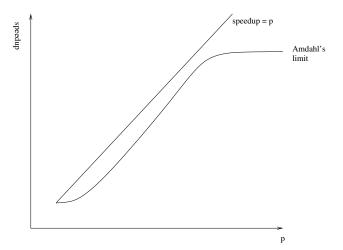
All Amdahl says is that an upper limit exists

Your program may not get anywhere close to that limit and often in real programs, does not

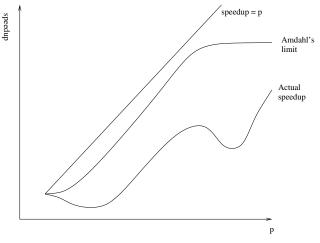
In real programs, there are many other factors that affect speedup, so that the speedup may well vary all over the place as p increases

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It can even decrease as *p* gets larger



Speedup in theory



Speedup in practice

To emphasize: all we know is that actual speedup is below Amdahl's limit

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Exercise Show that if $0 \le x \le 1$, then

$$\frac{1}{x+(1-x)/p} \le p$$

Exercise What is the maximum speedup of a program that is 100% sequential?

Amdahl's law is real: there is a natural limit on speedup for a given problem

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But there's another point of view

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Gustafson's Law (occasionally called *Gustafson-Barsis's Law*) gives us another limit

Speedup: Gustafson's Law

Suppose we have a problem of size *n*

$$\mathcal{S}_{p}(n) \leq rac{1}{x_n + (1-x_n)/p}$$

where $S_p(n)$ is the speedup on *p* processors for a problem of size *n*; x_n is the fraction of the computation spent sequentially

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So

$$S_{
ho}(\infty) \leq
ho$$

i.e., we now get a speedup limit that is the "perfect" speedup p — on an infinitely sized problem

Speedup: Amdahl's Law, Gustafson' Law

Both Amdahl and Gustafson are correct: they just apply to different cases of scaling

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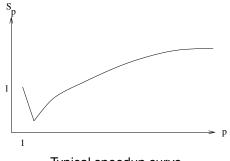
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This should convince you that even a simple measure like speedup can be problematic!

But it does re-emphasise the fact that parallelism is not about making things faster, but about making things larger

Analysis Speedup

Speedup is a simple measure, often proving that your parallel program is slower than it ought to be



Typical speedup curve

Sometimes it takes *p* to be surprisingly large before you even catch up with the uniprocessor time with $S_p = 1$ (sometimes never!)

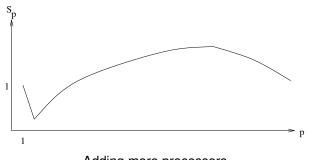
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But taking it further



Adding more processors

We might eventually find adding processors makes it slower!





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Of course, these are only typical behaviours: a given program may behave quite differently from all of this



Exercise Consider what might be the difference between a sequential implementation of something and a parallel implementation running on one processor

You will get used to seeing $\mathcal{S}_{p} < p$

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On the other hand, it is possible that $S_p > p$

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It is quite rare in real life, but it really can happen that a program runs more than p times as fast on p processors

This can happen for a variety of reasons, some technological, and some more philosophical





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Of course, this takes a certain kind of low-communication, easily dividable problem to work; and the right hardware



Note: modern CPUs tend to share cache across multiple cores, so it is unlikely *p* cores has *p* times as much cache



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(This helps with cache coherence!)

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Here is an example to illustrate the issue

We have bubblesort running on a uniprocessor: we wish to make it run on a parallel machine

One way of doing this is:

- split the data into equal halves
- bubblesort each half in parallel
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Depending on the number of processors we have, we can keep recursively dividing

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So bubblesorting the two halves (in parallel) takes time

$$(n/2)^2/2 + O(n/2) = n^2/8 + O(n)$$

Merging n values takes O(n), giving a total of

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Already superlinear!





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What is happening?



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Time to bubblesort halves: $2 \times (n^2/8 + O(n)) = n^2/4 + O(n)$; time to merge O(n); total $n^2/4 + O(n)$

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So we win even on a uniprocessor



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Merge sort has complexity $O(n \log n)$





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It may not always be possible to have a suitable parallel version of an algorithm: in such a case "speedup" is not meaningful

In most real cases we don't get this effect, but it's worth being aware that it can happen



Some people go further and define speedup as

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but this has its own problems, not least that we might not know the best possible sequential way of doing things



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And we now might be comparing two completely unrelated algorithms

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So, again, we are not really comparing like with like

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You would need to ensure each run had the same randomness to be properly comparable; or run many times and take an average time

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Other problems fare less well — in terms of speed — from parallelisation!