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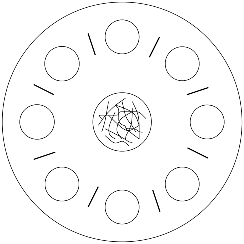
Often used to illustrate problems of resource contention in operating systems, it can be used to help understand problems in concurrency, too



Dining Philosophers

We have five philosophers wanting to eat spaghetti, but there are only five chopsticks to go round

The life of a philosopher is

- think
- sit
- take chopsticks
- eat
- drop chopsticks
- leave
- repeat

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Parallel Algorithms

Dining Philosophers

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If a chopstick is already in use, the philosopher must wait until it is free

This problem shows

Parallel Algorithms

Dining Philosophers

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- deadlock if all the philosophers sit down simultaneously and grab the left chopstick: they will all then have to wait on their right chopstick
- starvation, as four of the philosophers might conspire to keep out the fifth

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lock chopstick[5];
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Then philosopher *i* grabbing and dropping the chopsticks is

```
lock(chopstick[i]);
lock(chopstick[(i+1)%5]);
eat();
unlock(chopstick[(i+1)%5]);
unlock(chopstick[i]);
```

But, as we know, this can deadlock if all philosophers grab (say) the left chopstick simultaneously

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The classical solution is to have a counting semaphore, initialised to 4, to limit the number of simultaneously sitting philosophers

Parallel Algorithms

Dining Philosophers

```
lock chopstick[5];
place = make_counting_semaphore(4);
...
philosopher(int i) {
  while (1) {
    think();
    wait(place);
    lock(chopstick[i]);
    lock(chopstick[(i+1)%5]);
    eat();
    unlock(chopstick[(i+1)%5]);
    unlock(chopstick[i]);}
    signal(place);
  }
}
```

Exercise Prove this cannot deadlock

Exercise Think about fixing starvation

Exercise Solve the Dining Philosophers using monitors

Exercise Solve the Dining Philosophers using GCD

Parallel Algorithms

Sorting

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- divide data into two equal chunks
- recursively merge sort each half in parallel
- merge the two sorted lists together

For example, $n = 8$. The division is trivial, so we concentrate on the merge:

								t	p
3	1	4	1	5	9	2	6		

t is the time to merge sort that line; p the number of processors

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1	3	1	4	5	9	2	6	**2**	4

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1	1	3	4	2	5	6	9	**4**	2
1	1	2	3	4	5	6	9	**8**	1

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Parallel Algorithms
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1	1	3	4	2	5	6	9	**4**	2
1	1	2	3	4	5	6	9	**8**	1
							Total:	**14**	

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Total time is $T(n) = n + n/2 + n/4 + \cdots + 2 = 2n - 2 = O(n)$

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$$E = O(\log n/n)$$

The efficiency drops to 0 as n gets large

If we have just p processors, this becomes

$$T_p(n) = O\left(n + \frac{n}{p}\log\frac{n}{p}\right)$$

as we have sequential merge sorts of p chunks of size n/p, plus $(n/p)O(p) = O(n)$ steps to merge them in parallel

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Exercise Work this example through for yourself

So: for a fixed number of processors we can get good a speedup, but if we let the number of processors get large our relative speedup gets quite poor

Parallel Algorithms

Sorting

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Exercise Think about this result in the context of Amdahl and Gustafson

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- pick a value, the *pivot*, from the data
- partition the data into two chunks: values bigger than the pivot; values less than the pivot
- recursively quicksort the two chunks
- return the sorted lower chunk; the pivot; the sorted higher chunk

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Though we do need to join the sorted partitions back together

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As usual, quicksort relies on decent pivots: this translates directly to the need to get good load balancing of the sub-tasks

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But there doesn't seem to be a good way of parallelising it as the swaps in the heap creations and destructions need to pass in unpredictable ways through the entire dataset

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Clearly, an extension of the merge sort, it has very similar properties

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Some of these you need to be sorting upwards of 10^{22} items to be faster than simpler sorts with apparently worse complexities, like the *bitonic* sort, with time $O(\log^2 n)$

Parallel Algorithms

Sorting

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Exercise Go and read up on bitonic sort

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Particularly when you start to factor communications costs into your time complexities

Exercise It has been claimed that `MapReduce` can sort “a petabyte of data in a few hours”. Find out about how it does this

Exercise Related to sorting is the problem of finding the maximum value in a dataset. Discuss how this might be parallelised and its time complexity

Exercise Then find the middle value in a dataset

Parallel Algorithms

Searching

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Or, if multiple results are wanted, there can be a reduce step

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But linear search is far from a good sequential search

Again, we get a good speedup since we start from a poor place

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Sounds good? Well, consider the speedup for large n:

$$\begin{aligned} O(\log n/\log(n/p)) &= O(\log n/(\log n - \log p)) \\ &= O(1/(1 - \log p/\log n)) \\ &\to 1 \text{ as } n \to \infty \end{aligned}$$

Parallel Algorithms

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Here the problem is that tree search is so good that the benefit you get from spreading it across p processors is small, and gets smaller as the dataset increases in size

And these algorithms rely on everything being nice and uniform and randomly accessible and ignoring communications costs

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For example, if the searches cluster around the data on a single machine, we could write a sequential search that takes advantage of that fact, and our parallel search would not be much faster

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Again, we find that parallelism allows us to go bigger, rather than faster

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Reduction has a natural parallelisation using a tree

Parallel Algorithms

Reduction

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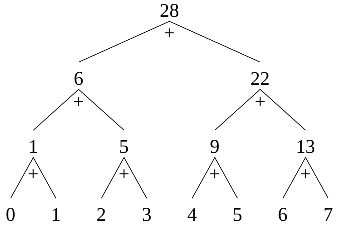
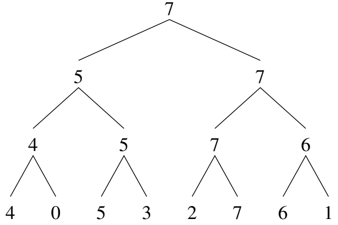


Tree reduction sum

Reducing a list of values using summation (read bottom up)

Next: parallel reduction

Reduction has a natural parallelisation using a tree



Tree reduction maximum

Reducing a list of values using maximum

This takes $O(\log n)$ steps to reduce n values, using $O(n)$ processors

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Sequential time: $n - 1$ operations, giving speedup

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This is not much less than n, as $\log n$ grows only slowly with n

Efficiency

$$E = O(1/\log n)$$

which slowly drops as n increases

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Parallel Algorithms

Reduction

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Time to reduce the chunks: $O(\log p)$

Total

$$O\left(\frac{n}{p} + \log p\right)$$

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For a fixed number of processors, you get a fixed bound on the speedup, but you will be using the hardware very efficiently as the dataset get large