Parallel Algorithms

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This is grain size, again

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$$((1 - 2) - 3) - 4 = -8$$

a *left* reduction

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Reduce the list $(1, 2, 3, 4)$ using $-$

Do we mean

$$((1 - 2) - 3) - 4 = -8$$

a *left* reduction

Or

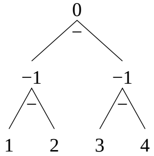
$$1 - (2 - (3 - 4)) = -2$$

a *right* reduction?

Parallel Algorithms

Reduction

And a tree reduction will give



Tree Reduction

Parallel Algorithms

Reduction

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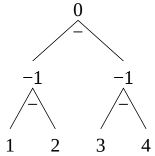

Tree Reduction

Or something else entirely depending on where the data ended up in the tree

The simple answer is not to do reductions using non-associative operations, even sequentially

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However, there are many useful reduction operations, including $+$, $*$, max, min, left$(a, b) = a$ and so on

Parallel Algorithms

Reduction

Reduction appears as an operation in many languages, e.g., JavaScript `array.reduce(op)` to reduce the `array` with the `op`:

`((array[0] op array[1]) op array[2]) op ...`

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```
((array[0] op array[1]) op array[2]) op ...
```

Thus amenable to automatic parallelisation, if the operation is associative and independent of the array (e.g., not if the `op` updates the array)

Parallel Algorithms

Prefix Scan

Closely related to reduction is the *prefix scan*: $(1, 2, 3, 4)$ with $+$ returns

$$(1, 3, 6, 10)$$

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The partial reductions, usually left associated

Parallel Algorithms

Prefix Scan

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For example, you can compute $3 + 4$ at the same time as $1 + 2$; and then $(1 + 2) + 3$ in parallel with $(1 + 2) + (3 + 4)$

Parallel Algorithms

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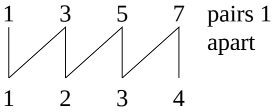
We can proceed in a tree-like sequence of combination of pairs of values

Parallel Algorithms

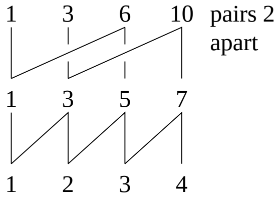
Prefix Scan



Prefix Scan 1 apart

Parallel Algorithms

Prefix Scan



Prefix Scan 2 apart

First step is to sum `array[i] = array[i] + array[i-1]` in parallel

Parallel Algorithms

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Parallel Algorithms

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Then double the distances:
`array[i] = array[i] + array[i-4]`

And so on, for $\log n$ steps on $O(n)$ processors: this gives us all the prefix sums, including the total reduction as the last element

When limited to p processors we can produce a scan in time

$$O\left(\frac{n}{p} + \log p\right)$$

Parallel Algorithms

Prefix Scan

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Scan has the same issues as reduce, namely data travel and associativity

Scan appears to give us more answers than reduce for the same amount of work!

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It's not: for a start, reduce uses at most $n/2$ processors, while scan uses up to $n - 1$

But more importantly, reduce halves the number of active processors in each step, while scan uses more processors more of the time. It uses $n - 2^r$ active processors in step r, so it *ends* with about $n/2$ active processors

Parallel Algorithms

Prefix Scan

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We can see that reduce has quite a lot of slack in parallel!

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Parallel Algorithms

Prefix Scan

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MPI includes several scan operations including `MPI_MAX`, `MPI_MIN`, `MP_SUM`, `MPI_PROD`, `MPI_LAND` (logical AND), `MPI_LOR` (logical OR)
amongst others

Parallel Algorithms

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Exercise Write a parallel prefix scan in OpenMP

Exercise In fact there is a better, work efficient, more complicated algorithm that only needs $n/2$ processors. Look it up

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Parallel Algorithms

FFT

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Parallel Algorithms

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If the input numbers represent a signal, the DFT values represent the constituent frequencies of that signal

$$y_k = \sum_{j=0}^{n-1} x_j e^{-2\pi ijk/n}, \text{ for } 0 \leq k < n$$

The n values x_i are input; the n values y_i are output

Parallel Algorithms

FFT

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Parallel Algorithms
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- each summation can be done as a tree, for a $\log n$-way parallelism
- taking total time $O(\log n)$ on $O(n^2)$ processors

But, instead let us look at a sequential divide and conquer version

This sum can be computed as presented: summing n values for each of n values y_k, thus taking time $O(n^2)$

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However, if n is even, then we get a nice recursive presentation by splitting the sum into evens and odds

Parallel Algorithms

FFT

$$
\begin{aligned}
y_k &= \sum_{j=0}^{n-1} x_j e^{-2\pi ijk/n} \\
&= \sum_{j=0}^{n/2-1} x_{2j} e^{-2\pi i(2j)k/n} + \sum_{j=0}^{n/2-1} x_{2j+1} e^{-2\pi i(2j+1)k/n} \\
&= \sum_{j=0}^{n/2-1} x_{2j} e^{-2\pi ijk/(n/2)} + e^{-2\pi ik/n} \sum_{j=0}^{n/2-1} x_{2j+1} e^{-2\pi ijk/(n/2)}
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Decomposition of Fourier Transform

Parallel Algorithms

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Decomposition of Fourier Transform

This is just two half-size DFTs

For n a power of 2 we can repeat recursively, leading to the *Fast Fourier Transform*, a way to implement the DFT

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Parallel Algorithms

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The FFT takes sequential time $O(n \log n)$, which is a huge improvement over $O(n^2)$; e.g., for $n = 1{,}000{,}000$, this is about 20,000,000 against 1,000,000,000,000

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But, for our purposes, we can see this as a simple divide and conquer, thus easily parallelisable

The parallelisation of the FFT works in a way very similar to what we have seen before and has complexity $O(\log n)$ on $O(n)$ processors, and $O(\log p + (n/p)\log(n/p))$ on p processors

Parallel Algorithms

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As the FFT is such an important algorithm, much has been written about it and its parallel variants, in particular matching it to the various kinds of hardware (SIMD, pipeline, shared memory, etc.)

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And So On

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Exercise Look some up!