Stability of multi-parameter solitons: asymptotic approach

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Received 4 May 1999; received in revised form 19 August 1999; accepted 21 October 1999

Communicated by A.C. Newell

Abstract

A general asymptotic approach to the stability problem of multi-parameter solitons in Hamiltonian systems has been developed. The presented approach gives an analytical criterion for oscillatory instability and also predicts novel stationary instability of solitons. Higher order approximations allow one to calculate corresponding eigenvalues with an arbitrary accuracy. It is also shown that asymptotic study of the soliton stability reduces to the calculation of a certain sequence of the determinants, where the famous determinant of the matrix consisting from the derivatives of the system invariants is just the first in the series. ©2000 Elsevier Science B.V. All rights reserved.

PACS: 05.45.Yv; 47.20.Ky; 42.65.Tg; 52.35.−g

Keywords: Soliton stability; Hamiltonian systems; Codimension-2 bifurcation

1. Introduction

Solitary waves (‘solitons’) can appear when an initial excitation applied to a medium is strong enough to cause nonlinear response. Formally, solitons are solutions of some nonlinear partial differential equations and their dynamics generally is a complex phenomenon, which can be described exactly only in the very special integrable situations [1]. The problems of soliton stability and instability induced dynamics in nonintegrable Hamiltonian models have paramount importance for the understanding of a wide range of physical phenomena covering such fields as propagation of electromagnetic, water and plasma waves, condensed matter physics and classical field theory [2–6,23,31,32]. Several analytical approaches to the stability problem are known. For instance, in the nearly integrable situations the perturbation theory based on the inverse scattering transform can be used [1,2]. Far from
The integrable limit, a variety of methods can be applied. Among them are the asymptotic stability theory [7,8], method of adiabatically varying soliton parameters [1,8–13,24,25], Lyapunov [5] and Evans [6,23] methods.

Generally, stability of a solitary wave in a Hamiltonian model can be lost due to bifurcations involving appearance of a positive eigenvalue (stationary instability) in the soliton spectrum or a pair of complex conjugate eigenvalues with positive real parts (oscillatory instability) [14]. Both types of these instabilities have been extensively studied in the different solitonic contexts proving their ubiquitousness and fundamental importance, see Refs. [6–10,15,16,23,26–28,31,32] and Refs. [6,11,16,17,23,29,30,35], respectively, for the stationary and oscillatory instabilities. In most of the known cases, the loss of stability is associated with the collisions of the purely imaginary eigenvalues corresponding to the so-called internal modes [20] of the soliton spectrum (see [6,23] for interesting exceptions).

Applying the above mentioned methods, it was shown that in many cases a threshold of stationary instability of multi-parameter solitons is given by the zero of the determinant of the Jacobi matrix $J_{ij} = \partial_{\kappa_j} Q_i$, where $\kappa_j$ are the soliton parameters and $Q_i$ are the associated motion integrals [3–11,15,23,26–28]. The condition $\det(J_{ij}) = 0$ is, in fact, the compatibility condition of the problem arising in the leading (zero) order of the asymptotic solution of the eigenvalue problem governing stability of the soliton [8,10,15,26–28]. To find expressions for the eigenvalues, it is necessary to proceed further and solve problems arising in higher (at least first) orders. Up to now, this was done only for the specific class of model equations having single parameter soliton families [7,8]. For stationary bifurcations of two-parameter solitons an adiabatic method has been applied in Refs. [10,11,13]. A linear approximation of this method, actually, gives an expression for eigenvalues, for more details see Section 4. However, all known developments of this method fail to give a criterion indicating transition to the oscillatory instability, i.e. instability with complex eigenvalues. It is also difficult to extend this method beyond its first order because of the rather involved calculations.

The purpose of this work is to formulate a general asymptotic approach to stability of multi-parameter solitons in Hamiltonian models, to show how it can be used to find expressions for the instability growth rates with arbitrary accuracy and to formulate a criterion for the oscillatory instability of solitons.

2. Model equations and symmetries

We will consider Hamiltonian equations in the form

$$\frac{i}{\hbar} \frac{\partial E_n}{\partial z} = \frac{\delta H}{\delta E_n^*}, \quad n = 1, 2, \ldots , N,$$

which describe a wide range of physical phenomena related with self-action and interaction of slowly varying wave envelopes in a variety of nonlinear media [3,9–12,15–17,24–30]; for a general review of the Hamiltonian formalism, see [18]. Here $E_n$ are complex fields, $z$ the propagation direction of the interacting waves, $x$ the coordinate characterizing dispersion or diffraction, $H = H(\partial_x E_n, E_n, \partial_x E_n^*, E_n^*)$ is the Hamiltonian and $^*$ means complex conjugation. We will assume that $H$ is invariant with respect to a set of $(L - 1)$ phase trans formations:

$$E_n \rightarrow E_n \exp(i\gamma_{nl}(\phi_l)), \quad l = 1, 2, \ldots , (L - 1),$$

$\phi_l$ are arbitrary real phases and $\gamma_{nl}$ are some constants. Because $H$ does not depend on $x$ explicitly, Eq. (1) is also invariant with respect to arbitrary translations along $x$.

$$E_n(x, z) \rightarrow E_n(x - x_0, z).$$
Symmetry properties (2) and (3) together with the Hamiltonian nature of our problem imply the presence of $L$ conserved quantities, see, e.g., [19], which are energy invariants

$$Q_l = \int dx \sum_{n=1}^{N} |\gamma_{nl}| E_n|^2, \quad l = 1, 2, \ldots, (L-1),$$

and momentum

$$Q_L = \frac{1}{2i} \int dx \sum_{n=1}^{N} (E_n^* \partial_t E_n - E_n \partial_t E_n^*).$$

Another important consequence of the invariances (2) and (3) is that a certain class of solutions of Eq. (1) can be sought in a form when $x_0$ and $l$ are linear functions of $z$, i.e. $x_0 = \kappa_L z$ and $l = \kappa'_L z$, then

$$E_n(x, z) = a_n (x - \kappa_L z) \exp \left( i \sum_{l=1}^{L-1} \gamma_{nl} \kappa_l z \right),$$

where $\{\kappa_l\}_{l=1}^{L-1}$ and $\kappa_L$ are real parameters characterizing, respectively, phase velocities of the interacting waves and the soliton group velocity. Functions $a_n(\tau)$ obey a system of the ordinary differential equations

$$(i\kappa_L \partial_t + a_n)a_n = -\delta H_a \delta a_n,$$

where $H_a \equiv H(\partial_t a_n, a_n, \partial_t a_n^*, a_n^*)$, $\tau = x - \kappa_L z$ and $a_n = \sum_{l=1}^{L-1} \gamma_{nl} \kappa_l$. We assume now that in a certain domain of the parameter space $(\kappa_1, \kappa_2, \ldots, \kappa_L)$, Eqs. (7) have a family of the solitary solutions such that $|a_n| \to 0$ for $\tau \to \pm \infty$.

3. Asymptotic stability analysis

To study stability of the solitons we seek solutions of Eq. (1) in the form

$$E_n = (a_n(\tau) + \epsilon_n(\tau, z)) \exp \left( i \sum_{l=1}^{L-1} \gamma_{nl} \kappa_l z \right),$$

where $\epsilon_n(\tau, z)$ are small complex perturbations. Linearizing Eq. (1) and assuming that $\epsilon_n(\tau, z) = \xi_n(\tau) e^{i \xi z}$, $\xi_n(\tau, z) = \xi_n(\tau) e^{i \xi z}$ we get the following nonselfadjoint eigenvalue problem (EVP)

$$i\lambda \xi = \hat{\mathcal{L}} \xi \equiv \begin{pmatrix} \hat{\mathcal{S}} & \hat{\mathcal{R}} \\ -\hat{\mathcal{R}}^* & -\hat{\mathcal{S}}^* \end{pmatrix} \xi,$$

where $\xi = (\xi_1, \ldots, \xi_N, \xi_{N+1}, \ldots, \xi_{2N})^T$, and $\hat{\mathcal{R}}$, $\hat{\mathcal{S}}$ are $N \times N$ matrix operators with elements given by

$$\delta_{nl} = \delta_{nl}(a_n + i\kappa_L \partial_t) + \frac{\delta^2 H_a}{\delta a_n^* \delta a_l}, \quad \hat{\mathcal{R}}_{nl} = -\frac{\delta^2 H_a}{\delta a_n^* \delta a_l^*},$$

here $\delta_{nl}$ is the Kronecker symbol. Note that the operator $\hat{\mathcal{S}}$ is selfadjoint, i.e. $\hat{\mathcal{S}} = \hat{\mathcal{S}}^*$, and $\hat{\mathcal{R}}$ a symmetric operator, i.e. $\hat{\mathcal{R}} = \hat{\mathcal{R}}^T$. 
To solve EVP (9) we apply an asymptotic theory. Known developments of this approach rely on expansion of the eigenvector \( \mathbf{\xi} \) into an asymptotic series near either neutral eigenmodes [7,8], i.e. zero-eigenvalue modes of the operator \( \hat{L} \), or modes of continuum [20]. The neutral modes can be generated by infinitesimal variations of the free parameters and thus can be presented as explicit functions of the soliton solution. At the same time continuum eigenmodes are explicitly known in the very rare, normally in integrable, situations [12,20,24]. This is an important fact which makes an asymptotic expansion near the neutral modes the very practical tool of the stability theory. However, as any approximate method, it has a certain limitation. Namely, it describes only eigenvalues \( \lambda \) corresponding to a specific class of the perturbations which in the zero approximation can be expressed as a linear superposition of the neutral eigenmodes. Thus, generally speaking, on the basis of this approach one can get only sufficient conditions for soliton instability or, in other words, necessary conditions for soliton stability.

Therefore, presence of other instabilities which can be captured only numerically can always be expected (see, e.g., [16,17]).

By infinitesimal variation of \( \phi_l \) and \( x_0 \) it can be shown that

\[
\mathbf{u}_l = (\gamma_l a_1, \ldots, \gamma_N a_N, -\gamma_l a_1^*, \ldots, -\gamma_N a_N^*)^T, \quad \mathbf{u}_N = \frac{\partial \mathbf{a}}{\partial \tau},
\]

\[
\mathbf{a} = (a_1, \ldots, a_N, a_1^*, \ldots, a_N^*)^T, \quad l = 1, \ldots, (L - 1)
\]

are neutral modes of \( \hat{L} \), i.e. \( \hat{L} \mathbf{u}_l = 0 \) \( (l = 1, \ldots, L) \). \( \hat{L} \) also has \( L \) associated vectors \( \mathbf{U}_l = \frac{\partial \mathbf{a}}{\partial k_l} \) such that \( \hat{L} \mathbf{U}_l = -\mathbf{u}_l \), \( l = 1, \ldots, L \).

It is straightforward to see that any solution of EVP (9) must obey \( L \) solvability conditions

\[
\langle \mathbf{w}_l | \lambda \mathbf{\xi} \rangle = 0, \quad l = 1, \ldots, L,
\]

where

\[
\mathbf{w}_l = (\gamma_l a_1, \ldots, \gamma_N a_N, \gamma_l a_1^*, \ldots, \gamma_N a_N^*)^T, \quad \mathbf{w}_N = \frac{\partial \mathbf{b}}{\partial \tau},
\]

\[
\mathbf{b} = (-a_1, \ldots, -a_N, a_1^*, \ldots, a_N^*)^T, \quad l = 1, \ldots, (L - 1).
\]

Associated vectors of \( \hat{L}^\dagger \) are \( \mathbf{W}_l = \frac{\partial \mathbf{b}}{\partial k_l} \) and they obey \( \hat{L}^\dagger \mathbf{W}_l = -\mathbf{w}_l \), \( l = 1, \ldots, L \).

The set of eigenfunctions of \( \hat{L} \) forms a natural basis for description of the soliton evolution in the presence of small perturbations of Eq. (1). These perturbations can be caused, e.g. by non-Hamiltonian corrections or random external driving of different physical origin, see e.g. [12,24,25,33,34]. Completeness of the eigenfunctions \( \hat{L} \)’s can be proved in some cases. This fact was recently used in the context of the dissipatively perturbed generalized nonlinear Schrödinger equation [25] to describe instabilities, including oscillatory ones, of the solitary waves, see also [33,34]. Our approach is different in two ways. First, we do not consider any perturbations of model equations, but only perturbations of a soliton itself. Second, we do not assume any particular form of \( \hat{L} \) and prior knowledge of its eigenfunctions, except generated by symmetries, but develop our theory to find those which make solitons unstable.

Close to the instability threshold it is natural to assume that \( |\lambda| \sim \epsilon \ll 1 \). As it was already discussed above, we will consider a special class of the perturbations which in the leading approximation can be presented as a linear combination of the neutral modes. Therefore, we seek an asymptotic solution of EVP (9) in the following form:

\[
\mathbf{\xi} = \sum_{m=0}^{\infty} e^{\epsilon m} \mathbf{\xi}_m(\tau), \quad \mathbf{\xi}_0 = \sum_{l=1}^{L} C_l \mathbf{u}_l,
\]
where constants $C_l$ and vector-functions $\xi_{m>0}$ have to be defined. Here and below $l = 1, 2, \ldots, L$. Substitution of (11) into EVP (9) gives a recurrent system of equations for $\xi_m$

$$\xi_{m>0} = \left[ \frac{i\lambda}{\epsilon} \hat{L}^{-1} \right]^m \xi_0. \tag{12}$$

Substituting (11) and (12) into conditions (10) one will find the homogeneous system of the $L$ linear algebraic equations

$$\lambda^2 \left\{ \sum_{m=0}^{\infty} (-\lambda^2)^m \hat{L}^{-2m} \sum_{l=1}^{L} C_l U_l \right\} = 0 \tag{13}$$

for $L$ unknown constants $C_l$. System (13) has a nontrivial polynomial with respect to $\lambda^2$, which, in fact, is the asymptotic expansion of the Evans function [6,23]. Zeros of this polynomial define spectrum of the solitary wave linked with the chosen class of perturbations. Thus, the equation specifying eigenvalues $\lambda$ is

$$\lambda^{2L} \sum_{j=0}^{\infty} (-\lambda^2)^j D_j = 0, \tag{14}$$

where $D_j$ are real constants. Eq. (14) always has zero roots of the $2L$-order. It indicates that each of the zero eigenvalues corresponding to the neutral modes $u_l$ is doubly degenerate. This degeneracy originates from the presence of the associated vectors $U_l$.

To write explicit expressions for $D_j$ it will be convenient to introduce vectors $M_l^{(m)} = (M_{l1}^{(m)}, \ldots, M_{L}^{(m)})$, where,

$$M_l^{(m)} = \{W_l | \hat{L}^{2m} U_l^\prime\}, \quad m = 0, 1, \ldots, \infty.$$

Now each $D_j$ can be presented as

$$D_j = \sum_{m_1 + \ldots + m_L = j} D(M_{l1}^{(m_1)}, \ldots, M_{L}^{(m_L)}), \tag{15}$$

where $D(M_{l1}^{(m_1)}, \ldots, M_{L}^{(m_L)})$ is the determinant of the $L \times L$ matrix consisting of the rows $M_l^{(m_l)}$ and the sum is taken over all such combinations of $(m_1, \ldots, m_L)$ such that $\sum_{l=1}^{L} m_l = j$. $M_{l0}$ can be readily expressed via derivatives of the conserved quantities with respect to the soliton parameters:

$$M_{l0}^{(m)} = \frac{\partial Q_l}{\partial k_{l0}} \tag{16}$$

and practical calculations of $M_{l0}$ for $m > 0$ can be simplified: $M_{l0}^{(m)} = -(W_l | \hat{L}^{1-2m} U_l^\prime)$. Note that in most of the cases the solitary solution itself can be found only numerically using any of the well established methods for solving the nonlinear ODEs. Recurrent calculations of $\hat{L}^{1-2m} U_l$ can be readily reduced to the numerically even simpler problem of solving the linear inhomogeneous ODEs.

Because $|\lambda|$ was assumed to be small, Eq. (14) has an asymptotic character. Therefore, to make it work, some additional assumptions must be made about orders of $D_j$. If these assumptions are satisfied, then Eq. (14) describes correctly the soliton spectrum and predicts bifurcations of the soliton. The corresponding eigenvalues can be found
using Eq. (14) with any degree of accuracy. For example, let us assume that \( D_0 \sim \epsilon^2 \) and \( D_{j>0} \sim O(1) \). Then, presenting \( \lambda^2 \) as

\[
\lambda^2 = \epsilon^2 \sum_{j=0}^{\infty} \xi_j, \quad \xi_j \sim \epsilon^{2j},
\]

in the first order (14) gives a linear equation for \( \xi_0 \),

\[
D_0 - \epsilon^2 \xi_0 D_1 = 0,
\]

which indicates a threshold of the stationary bifurcation at \( D_0 = 0 \). This is precisely the condition \( \det(J_{ij}) = 0 \) discussed in the introduction. Continuing to the next order, one obtains

\[
\lambda^2 = \frac{D_0}{D_1} \left( 1 - \frac{D_0 D_2}{D_1^2} + O(\epsilon^4) \right).
\]

If \( D_1 \sim \epsilon^2 \), then the asymptotic expression (19) fails. To have a balanced equation for \( \xi_0 \), we must now assume that \( D_0 \sim \epsilon^4 \). However, in such a case Eq. (18) for \( \xi_0 \) changes from linear to quadratic

\[
D_0 - \epsilon^2 \xi_0 D_1 + \epsilon^4 \xi_0^2 D_2 = 0.
\]

Eq. (20) gives two threshold conditions \( D_0 = 0 \) and \( D_1^2 = 4D_0D_2 \) (see Fig. 1). The latter condition indicates the onset of the oscillatory instability for

\[
D_1^2 < 4D_0D_2.
\]

Thus we have formulated an analytic criterion for the oscillatory instability. It is also clear that the point \( D_{0,1} = 0 \) is a source for the novel stationary instability, see the rightmost region \( D_1^2 > 4D_0D_2, \ D_1 > 0 \) in Fig. 1, where an eigenvalue which is positive throughout this region cannot be predicted by Eq. (19).

It follows by recurrence that if \( D_{j' > 0} \sim \epsilon^2 \) then to have a balanced equation for \( \xi_0 \) we must assume that \( D_{j+j'} \sim \epsilon^{2(1+j')} \). In other words, asymptotic expansion near the neutral modes can only describe the soliton spectrum in regions of the parameter space which are close to the codimension-(\( j' + 1 \)) bifurcation. If \( j' = 0 \), then only one condition must be satisfied and our asymptotic approach predicts presence of either two purely imaginary or two purely real eigenvalues, which can collide at zero. If \( j' = 1 \) then two conditions must be satisfied and the asymptotic approach predicts presence of two pairs of eigenvalues which can be real, imaginary or complex. For each further \( j' \) two new eigenvalues come into play.
4. Discussion

General formulae (14) and (15) giving soliton eigenvalues with any degree of accuracy and criterion for the oscillatory instability (21) are the main novel results of this work. Expressions for the eigenvalues near the stationary instability threshold, analogues of the formula $\lambda^2 = D_0/D_1 + \ldots$, have been earlier obtained in a number of papers. It is instructive now to give explicit expressions for $D_j$ in the two simplest situations of one- and two-parameter solitons and to compare them with previously reported results. For the one-parameter solitons: $D_0 = \partial Q_1 / \partial k_1, D_1 = -\langle W_1 | \hat{L}^{-1} U_1 \rangle$ and $D_2 = -\langle W_1 | \hat{L}^{-2} U_1 \rangle$. Using these formulae one can show that in the case when $D_1 \sim O(1)$ the first term in Eq. (19) gives the same expression for $\lambda^2$ which was obtained in Refs. [7–9], where the generalised nonlinear Schrödinger equation [7,9] and equations describing propagation in quadratically nonlinear media [8] have been investigated. If $D_1 D_2 > 0$ then it can be concluded that the second term in Eq. (19) indicates saturation of the growth rate when the distance from the instability threshold, $D_0 = 0$, grows, which agrees with numerical results [8,16].

For two-parameter solitons

\[ D_0 = \begin{vmatrix} \partial Q_1 / \partial k_1 & \partial Q_1 / \partial k_2 \\ \partial Q_2 / \partial k_1 & \partial Q_2 / \partial k_2 \\ M_{11}^{(1)} & M_{12}^{(1)} \end{vmatrix}, \]

\[ D_1 = \begin{vmatrix} \partial Q_1 / \partial k_1 & \partial Q_1 / \partial k_2 \\ \partial Q_2 / \partial k_1 & \partial Q_2 / \partial k_2 \\ M_{11}^{(1)} & M_{12}^{(1)} \end{vmatrix} + \begin{vmatrix} M_{11}^{(2)} & M_{12}^{(2)} \\ M_{21}^{(2)} & M_{22}^{(2)} \end{vmatrix}, \]

\[ D_2 = \begin{vmatrix} \partial Q_1 / \partial k_1 & \partial Q_1 / \partial k_2 \\ \partial Q_2 / \partial k_1 & \partial Q_2 / \partial k_2 \\ M_{11}^{(1)} & M_{12}^{(1)} \end{vmatrix} + \begin{vmatrix} M_{11}^{(2)} & M_{12}^{(2)} \\ M_{21}^{(2)} & M_{22}^{(2)} \end{vmatrix} + \begin{vmatrix} M_{11}^{(3)} & M_{12}^{(3)} \\ M_{21}^{(3)} & M_{22}^{(3)} \end{vmatrix}. \]

The threshold condition $D_0 = 0$ has been previously found for two-parameter solitons in different physical contexts [10,11]. However, derivation of an accurate expression for the soliton eigenvalues near this threshold has remained a controversial problem. Indeed, comparing eigenvalues given by Eqs. (19) and (22)–(24) and eigenvalues which can be calculated from the ordinary differential equations for soliton parameters presented in [10,11] one will discover that results are slightly different. ²

Among open problems, derivation of finite-dimensional normal forms describing dynamical evolution of the soliton parameters near the oscillatory instability threshold may be mentioned. A guideline for this work can be theory of Iooss [21,22] for the normal forms of the reversible ordinary differential equations [22] in the vicinity of the codimension-2 bifurcation, which is an equivalent of the point $D_0 = D_1 = 0$. The simplest case of the codimension-1 stationary instability, $D_0 = 0$, has only one homoclinic orbit separating regions of the periodic oscillations from spreading or collapse [9,13]. The vicinity of the codimension-2 point can contain the very rich dynamics, including multiple homoclinic orbits and stochastic regimes.

² For example, reading p. 86 after Eq. (10) of Ref. [11] one can find that $\lambda^2 = -J/\langle m \partial \delta_x Q_0 \rangle$, where $J$ is exactly $D_0$ given by Eq. (22) and the denominator can be rewritten in our notations as $-(\delta_x Q_1)^3 / \delta_x Q_1 M_{11}^{(1)} - M_{12}^{(1)} \delta_x Q_1 + M_{12}^{(1)} \delta_x Q_2 + M_{22}^{(1)} \delta_x Q_1$. One should expect that the last expression is equal to $-D_1$, but it is not, due to its first term. The source of this disagreement seems to be hidden in the fact that authors of Refs. [10,11] develop adiabatic theory of the order $\epsilon^2$ using the ratio between time derivatives of the soliton parameters obtained from the solvability condition valid only to the order $\epsilon$, see text after Eq. (9) in [11] and Eqs. (7) and (10) in [10].
5. Summary

A general form of the asymptotic approach to the stability problem of multi-parameter solitons in Hamiltonian systems has been developed. It has been shown that the asymptotic study of the soliton stability reduces to the calculation of a certain sequence of determinants, where the famous determinant of the matrix consisting from the derivatives of the system invariants with respect to the soliton parameters [4,5,10] is just the first in the series. Knowledge of these determinants allows one to calculate eigenvalues governing soliton instability with arbitrary accuracy. The most important consequence is that the presented approach gives a analytical criterion for the oscillatory instability of solitons in Hamiltonian systems.

Acknowledgements

Author acknowledges useful remarks from W.J. Firth, T. Kapitula and D.E. Pelinovsky and financial support from the Royal Society of Edinburgh and British Petroleum.

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