Energy of internal modes of nonlinear waves and complex frequencies due to symmetry breaking

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Considering a class of Hamiltonian systems, it is demonstrated that energy of the internal modes with real frequencies supported by nonlinear waves and appearing due to perturbations breaking a continuous symmetry has its sign determined by the symmetry itself, independently of the nature of the perturbations. In particular, it is shown that negative energy modes emerge as a result of the breaking of the phase symmetry in the perturbed nonlinear-Schrödinger equation. An expression for energy of the Vakhitov-Kolokolov internal modes is also derived. Comparative analysis of the energy signs of the internal modes in these two cases explains the ubiquity of instabilities with complex frequencies of solitary and continuous waves in systems with broken phase symmetry.

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Studies of the internal modes supported by nonlinear waves of different physical origin have attracted a great deal of attention over recent decades. The presence of these modes is responsible, e.g., for persistent oscillatory dynamics and is directly linked to a most important problem of nonlinear dynamics—stability of an equilibrium [2–5].

It is clear that internal modes being excited by imperfections in initial conditions should carry certain energy. The amount of energy carried by a particular internal mode depends on the strength and geometry of the initial perturbations and cannot be considered as an intrinsic characteristic of the system and its equilibria. On the contrary, the sign of the energy of an excited mode does not generally depend on the magnitude of the perturbation. Therefore, analysis of these signs, which are often referred in mathematical literature as signatures or Krein-signs, can be expected to highlight important qualitative information, for reviews see [4,5].

The notion of energy of internal modes is similar to the quantum-mechanical definition of energy through the eigenvalues of the Hamiltonian and it can be considered as a generalization of the latter to the class of non-self-adjoint (non-Hermitian), but still Hamiltonian problems. One of the striking predictions of the energy analysis is that frequency resonances of the internal modes with opposite energy signs (opposite signatures), is the most typical scenario leading to the appearance of complex frequencies in the spectrum of an equilibrium [4,5]. This type of instability can be interpreted as mutual annihilation of a particle with its antiparticle. Analysis of the energy of the internal modes has been used, e.g., to interpret instabilities of plasmas [4, 6] and water waves [4,7], vortices in superfluids [8,9], and optical solitons [10] and breathers [2].

One of the key problems, which has remained unaddressed so far, is whether there are any universal mechanisms governing the appearance of the modes with opposite energy signs leading to their subsequent resonances. In this work we will demonstrate that one of such mechanisms is the breaking of the continuous symmetry of the governing equations. Our analysis will be fairly general and supported by consideration of such ubiquitous examples as the perturbed nonlinear Schrödinger (NLS) equation and degenerate three wave mixing, which are applicable to many problems in optics, fluid dynamics, plasma physics, and other fields.

We start by considering a Hamiltonian system in the form

$$i \partial_t \psi + \frac{\delta H}{\delta \psi^*} = 0,$$

(1)

describing the evolution of $N$ interacting waves with complex amplitudes $\psi_j$, $\psi = (\psi_1, \ldots, \psi_N, \psi_1^*, \ldots, \psi_N^*)^T$. $\eta = \text{diag}(-I, I)$ is the structure matrix, $I$ is the $N \times N$ identity operator, $H$ is the energy functional (or Hamiltonian), and $\delta H/\delta \psi^* = (\delta H/\delta \psi_1^*, \ldots, \delta H/\delta \psi_N^*)^T$. Below we often use the identity $\hat{\eta} \eta = \text{diag}(I, I)$.

To study the stability of an equilibrium state $\psi_e$ found from the condition $\delta H/\delta \psi^* = 0$, we make the substitution $\psi = \psi_e + \tilde{a}$ into $H$ and disregard all terms of order higher than second in the components of $\tilde{a}$. This procedure leaves us with linear equations describing evolution of the perturbation vector $\tilde{a}$:

$$i \partial_t \tilde{a} = -\eta \frac{\delta B}{\delta \tilde{a}^*} = -\hat{B} \tilde{a},$$

(2)

where $B = \frac{1}{2} \langle \tilde{a} | \hat{B} | \tilde{a} \rangle$ is the quadratic part of the Hamiltonian, $\hat{B}$ is the linear $2N \times 2N$ self-adjoint operator

$$\hat{B} = \begin{bmatrix} \hat{S} & \hat{R} \\ \hat{R}^* & \hat{S}^* \end{bmatrix}, \quad \hat{S}_{ij} = \frac{\delta^2 H}{\delta \psi_i^* \delta \psi_j}, \quad \hat{R}_{ij} = \frac{\delta^2 H}{\delta \psi_i^* \delta \psi_j^*},$$

(3)

and $\hat{S}$, $\hat{R}$ are $N \times N$ matrix operators with elements $\hat{s}_{ij}$, $\hat{r}_{ij}$. $\langle \ldots | \ldots \rangle$ defines scalar product, $\langle \hat{f} | \hat{g} \rangle = \int dV \sum f_i^* g_i$. Second variations in Eqs. (3) are taken for $\psi = \psi_e$. Assuming $\tilde{a} = \xi \exp(i\omega t) + \hat{g} \exp(-i\omega^* t)$, where $\tau$ is the transposition matrix, $\hat{\tau} = \text{antidiag}(\hat{I}, \hat{I})$, we derive the non-self-adjoint eigenvalue problem
\[ \hat{\eta} B \xi_n = \omega_n \xi_n , \]  

where \( \xi_n \) are eigen- or normal modes of the perturbations and \( \omega_n \) are their eigenfrequencies. Internal modes are eigenmodes with real frequencies belonging to the discrete spectrum.

Perturbing an equilibrium state \( \hat{\phi}_\epsilon \), one generally either adds or subtracts energy. Assuming that \( \hat{\phi}(t=0) = \xi_n + \tau \xi_n^* \), we can define the energy \( \epsilon_n \) of the normal perturbation as

\[ \epsilon_n = \langle \xi_n | \hat{B} | \xi_n \rangle = \omega_n \langle \xi_n | \hat{\eta} | \xi_n \rangle . \]

\( \epsilon_n \) is simply the difference between the Hamiltonians for the perturbed and unperturbed equilibria.

Significant insight into Eq. (5) can be achieved, using the biorthogonality of the eigenmodes of \( \hat{\eta} B \) and of the adjoint operator \( \hat{B} \hat{\eta} \), i.e., \( \langle \xi_n | \tilde{\nu}_n \rangle = 0 \), where \( n \neq n' \) and \( \tilde{\nu}_n \) obeys \( (\hat{\eta} B) \tilde{\nu}_n = B \hat{\eta} \tilde{\nu}_n = \omega_n \tilde{\nu}_n \). If \( \omega_n \) is real, then \( \tilde{\nu}_n = \xi_n^* \) and if \( \omega_n \) is complex or imaginary, then \( \tilde{\nu}_n = \xi_n^* \). In the latter case, \( \tilde{\nu}_n \) is also an eigenmode of \( \hat{B} \hat{\eta} \), but with eigenfrequency \( \omega_n \). Hence if \( \text{Im} \omega_n \neq 0 \), then biorthogonality implies \( \langle \xi_n | \tilde{\nu}_n \rangle = \epsilon_n = 0 \), i.e., modes with complex or imaginary frequencies carry zero energy, see also [5,8]. For modes with real frequencies, the magnitude, but not the sign of \( \epsilon_n \), can be scaled. However, it is more convenient for us to continue without any normalization, keeping in mind that \( \text{sgn}(\epsilon_n) \), not \( |\epsilon_n| \), has paramount importance. For more details of the interplay between the biorthogonality and energy signs, see [8].

We assume now that the equilibrium under consideration is spectrally stable, i.e., it has real spectrum, and that \( H \) can be present in the form \( H = H_0 + \mu H_{sb} \). \( H_0 \) is invariant with respect to a continuous one-parameter symmetry group \( g_\phi \), and \( H_{sb} \) describes corrections breaking this symmetry. Note that previous papers exploring the energy sign ideas in the context of the stability of solitary waves, see, e.g., [2,10] and references therein, have not considered the problem of the energy signs of the internal modes emerging due to perturbations breaking the continuous symmetry group of the Hamiltonian. The parameter \( \mu \) characterizes the strength of the symmetry breaking perturbations and it is assumed to be a small parameter in our analysis. \( \hat{B} \) also can be presented in the form \( \hat{B}_0 + \mu \hat{B}_{sb} \), where \( \hat{B}_{sb} \) contains the corrections to the Hamiltonian and the equilibrium. It is an important fact that the energy sign of the internal modes preserves as parameters vary as long as there is no frequency resonance with another mode, in which case energy can become zero [5]. Thus if the energy sign is found for \( \mu \ll 1 \), it will remain the same even outside the region of validity of the asymptotic expansion.

We also assume that the infinitesimal transformation \( g_{\delta \phi} \) applied to the equilibrium \( \hat{\phi}_\epsilon \) generates a Goldstone or neutral mode \( \hat{U}_0 \) such that \( \hat{B}_0 \hat{U}_0 = 0 \). It is obvious that \( \langle \hat{U}_0 | \hat{\eta} \hat{U}_0 \rangle = 0 \). Therefore, the generalized eigenvector \( \hat{U}_1 \), defined as the solution of \( \hat{\eta} B \hat{U}_1 = \hat{U}_0 \), exists so that the zero eigenfrequency of \( \hat{\eta} B \) with eigenvector \( \hat{U}_0 \) has algebraic multiplicity 2.

We aim to calculate the frequencies and energy signs of the eigenmodes of \( \hat{\eta} B \), replacing \( \hat{U}_0 \) under the action of the symmetry breaking perturbations. In order to do this we expand the solution of Eq. (4) as an asymptotic series in \( \mu \) assuming that \( \omega \sim \sqrt{\mu} \). Keeping the first two terms we find that

\[ \xi = \hat{U}_0 + \omega \hat{U}_1 + \ldots \]

Next, we substitute Eq. (6) into the orthogonality condition

\[ \mu(\hat{\eta} B_{sb} \xi | \hat{\eta} \hat{U}_0) = \omega(\xi | \hat{\eta} \hat{U}_0) , \]

which is derived by multiplying Eq. (4) by \( \hat{\eta} \hat{U}_0 \). To leading order we find that frequencies of the modes appearing due to symmetry breaking can be found from

\[ \omega_{sb}^2 (\hat{U}_1 | \hat{\eta} \hat{U}_0) = \mu (B_{sb} \hat{U}_0 | \hat{U}_0) + O(\mu^{3/2}) \]

and the corresponding energy is

\[ \epsilon_{sb} = \omega_{sb} (\omega_{sb} + \omega_{n}^*) (\hat{U}_1 | \hat{\eta} \hat{U}_0) + O(\mu) . \]

All particular features of the perturbations are hidden inside \( \omega_{sb} \) and their role is only in determining whether \( \omega \) is real or imaginary. If \( \text{Im} \omega_{sb} = 0 \), then \( \epsilon_{sb} \sim \omega_{n}^* \), and therefore the energy sign of the emerging spectrally stable internal modes is independent from the symmetry-breaking perturbations \( \hat{\eta} B_{sb} \), but is solely determined by the properties of the broken symmetry reflected in the structure of the operator \( \hat{\eta} B_0 \) and its eigenmodes. Thus,

\[ \text{sgn}(\epsilon_{sb}) = \text{sgn}(\hat{U}_1 | \hat{\eta} \hat{U}_0) . \]

Note that the latter result for the energy sign is valid for arbitrary large \( \mu \) providing that no frequency resonances with other internal modes take place. If \( \text{Re} \omega_{sb} = 0 \) and \( \text{Im} \omega_{sb} \neq 0 \), i.e., symmetry breaking leads to spectral instability, then \( \epsilon_{sb} = 0 \).

To illustrate applications and further develop the above ideas, we consider the fundamental examples of the two-dimensional (2D) NLS equation and three wave mixing. 2D NLS with generalized nonlinearity, can be derived from the Hamiltonian

\[ H_0 = \frac{1}{2} \int dx dy \left( |\nabla \psi|^2 + |\psi|^2 - \int d|\psi|^2 d I f(I) \right) . \]

We assume below that \( 0 < \theta < \max_{f(I)} \) and \( \delta_{f,I} > 0 \), which ensures existence of bright solitary solutions [3,11]. The symmetry of interest is

\[ \psi \rightarrow \psi e^{i \phi} , \]

with the corresponding Goldstone mode \( \hat{U}_0 = (\psi_0, -\psi_0)^T \). Here \( \psi_0 \) is a real function obeying \( \nabla^2 \psi_0 = (\theta - f(\psi_0^2))\psi_0 \),
which characterizes the transverse profile of the soliton. The
generalized eigenvector associated with \( \hat{U}_0 \) is \( \hat{U}_1 = \partial_\theta (\psi_0, \psi_0^*) \).

Typical perturbations breaking symmetry (12) and pre-
serving the Hamiltonian structure of the equations are either external,

\[
H_{sb}^{(c)} = - \int \int dx \, d\psi \{ \psi \ast \psi + \psi \}
\]  

(13)
or parametric,

\[
H_{sb}^{(p)} = - \frac{1}{2} \int \int dx \, d\psi \{ \psi^* \psi + \psi \psi^* \}
\]  

(14)

forcing. In the context of nonlinear optics, Hamiltonians
\( H_0 + \mu H_{sb}^{(c,p)} \) describe, respectively, nonlinear optical cavi-
ties with intensity dependent purely dispersive nonlinearity
excited by an external monochromatic pump [12] and intra-
cavity parametric wave mixing when the pump field is far
off-resonance or not resonated at all [13]. The parameter \( \theta \)
has physical meaning as the frequency detuning of the field
from the cavity resonance. The symmetry breaking perturba-
tions (13) and (14) also arise in the plasma physics [14],
dynamics of fluids [15], spin waves in ferromagnets and
other contexts [16].

It follows from Eq. (10) that the pair of the zero-energy
modes \( \hat{U}_0 \) transforms into the internal modes carrying energy
with signs given by

\[
\text{sgn}(\varepsilon_{ib}) = \text{sgn}(-\partial_\theta Q),
\]  

(15)

where \( Q = \int \int dx \, d\psi \psi_0^* \) is the power or number of particles
invariant. Thus if the Vakhitov-Kolokolov (VK) stability cri-
terion [11]

\[
-\langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle = \partial_\theta Q > 0
\]  

(16)
is satisfied for \( \mu = 0 \), then the breaking of the phase symmetry
leads to the emergence of negative-energy modes. Equations
(9), (10) and (16) represent main conceptually different
results of this work. The rest of the paper is largely devoted
to the illustration of their applicability.

It is clear that Eq. (8) loses its validity when \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle \)
either changes sign or is simply close to zero. Therefore, the
perturbation expansion capturing symmetry breaking effects
has to be modified in the vicinity of such a point. However,
before proceeding with these modifications, it is instructive
to consider the situation, when \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle \) is small, but no symmetry breaking is present.

The system \( \hat{\eta} \hat{B} \hat{U}_3 = \hat{U}_1 \) is solvable if \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle = 0 \).

Then equation \( \hat{\eta} \hat{B} \hat{U}_3 = \hat{U}_2 \) is also solvable without any ad-
tional constraint, simply because \( \langle \hat{U}_2 | \hat{\eta} \hat{U}_0 \rangle = 0 \). Therefore
the zero eigenfrequency with eigenvector \( \hat{U}_0 \) has now al-
gebraic multiplicity 4. If we assume \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle \sim \omega^2 \), then
the solution of Eq. (4) with \( \hat{B} = \hat{B}_0 \), up to the fourth order, is

\[
\hat{\xi} = \hat{U}_0 + \omega \hat{U}_1 + \omega^2 \hat{U}_2 + \omega^3 \hat{U}_3 + \mathcal{O}(\omega^4).
\]  

(17)

Substituting Eq. (17) into the right-hand side of the orthogo-
nality condition (7), we find that \( \omega_{nk}^2 \langle \hat{U}_2 | \hat{\eta} \hat{U}_1 \rangle + \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle = 0 \). Thus, when \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle \) passes through zero,
then two purely real eigenvalues pass through zero and move
along the imaginary axis. The above consideration is just
another way to prove validity of the VK criterion (16).

Using expansion (17), we can calculate energy \( \varepsilon_{ib} \) carried
by the internal modes causing the VK instability. Substituting
Eq. (17) into Eq. (5), we find

\[
\varepsilon_{ib} = -2 \omega_{nk} (\omega_{nk}^3 + \omega_{nk}^3) \langle \hat{U}_2 | \hat{\eta} \hat{U}_1 \rangle + \mathcal{O}(\omega_{nk}^2). \]  

(18)
The criterion (16) implies practically the spectral stability of
the solitary waves in many problems, despite the fact that its
sufficiency can be rigorously proved only in a limited num-err of cases [17]. Stability necessary requires \( \omega_{nk}^2 > 0 \), which
together with (16), readily implies

\[
\text{sgn}(\varepsilon_{ib}) = \text{sgn}(\hat{U}_1 | \hat{\eta} \hat{U}_1 ) > 0. \]  

(19)

In fact, positivity of the energy carried by the VK modes also
readily follows from the variational methods of the proof of
the VK criterion given in [17].

It follows from the preceding that if perturbations break-
ing the phase symmetry have been added into a system with
a VK threshold, then coexistence of the modes with positive
and negative energies and subsequent complex-frequency
instability should be expected. A famous example of a system
with four-fold degeneracy of \( \hat{U}_0 \) is provided by the ground-
state solitary wave of the Hamiltonian (11) with \( f(I) = I \).

Any perturbation consisting of saturating nonlinearity, the
simplest of which is \( f(I) = -I = -B^2 \), \( 0 < B < 1 \), replaces the
critical four-fold degeneracy by a noncritical double degene-
ricity and leads to emergence of the low-frequency positive-
energy VK modes. An interesting property of 2D NLS with
\( f = I \) is that the second generalized eigenvector can be found
explicitly [3]: \( \hat{U}_2 = 1/B^2 \hat{\theta} \hat{U}_0 \).

To verify that symmetry breaking close to the VK thresh-
hold \( \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle \sim \omega^2 \) does indeed lead to a complex-frequency
instability in driven 2D NLS, we use again perturbations
(13), (14), and expansion (17). Assuming that \( \mu^{1/4} \sim \omega \),
we find that the orthogonality condition (7) gives

\[
\langle \hat{B}_{ib}^{(c,p)} | \hat{U}_0 \rangle = \omega^4 \langle \hat{U}_2 | \hat{\eta} \hat{U}_1 \rangle + \omega^2 \langle \hat{U}_1 | \hat{\eta} \hat{U}_0 \rangle. \]  

(20)

Then, taking \( \beta \sim \sqrt{\mu} \), we show that the solitary solutions
themselves are

\[
\psi^{(c,p)} = e^{i\xi} e^{i\zeta}, \quad \{ \sqrt{\theta} \Psi_0 + \beta \Psi_2^0 \}^3 + (\mu/\Psi_0 + 3 \Psi_0^2) + (\mu/\Psi_0 + \rho \Psi_0^2)(\Psi_0 + \rho \Psi_0^2) \}
\]  

(21)

where \( \Psi_2^0 \) and \( \rho \) are the phases locked by
the perturbations \( \psi^{(c,p)} \), \( \psi^{(c,s)} = \pi, \psi^{(c,c)} = \pi/2 \), i.e., for\n each of the cases the phase can be locked to two physically
distinct values. Function \( \Psi_3 \) is irrelevant for our purposes
and \( \Psi_0, \Psi_2^0 \) are solutions of the parameter independent
equations \( \hat{D} \Psi_0 \) and \( \hat{D} \Psi_2^0 \) with \( \hat{D} = \hat{d} \Psi_0 \) and \( \hat{D} = \hat{d} \Psi_2^0 \) with \( \hat{d} = \hat{d}_0 \). Numerically
solving the latter, we find that

\[
\langle \hat{B}_{ib}^{(c,p)} | \hat{U}_0 \rangle = \pm 29.8 \mu, \quad \langle \hat{B}_{ib}^{(c,p)} | \hat{U}_0 \rangle = \pm 29.8 \mu. \]  

(22)

\[
\langle \hat{B}_{ib}^{(c,s)} | \hat{U}_0 \rangle = \pm 0.553 \pi/\theta^2 \]  

(23)

\[
\langle \hat{B}_{ib}^{(c,c)} | \hat{U}_0 \rangle = \pm 0.553 \pi/\theta^2 \]  

(24)
and \( \langle \tilde{U} | \tilde{\eta} \tilde{U}_0 \rangle = -15.084 \pi \beta \). Now it can be easily shown that soliton branches with \( \psi^{(\rho)} = \psi^{(\rho)} \) are stable when \( \mu \) deviates from zero, but lose their stability due to appearance of the complex frequencies, when \( \mu \) exceeds threshold values \( \mu^{(\rho)}_{1b} = 3.5 \theta^{2} \beta^{2} \) and \( \mu^{(\rho)}_{2b} = 10.7 \theta^{2} \beta^{2} \). The branches \( \psi^{(\rho)} \) have a zero-energy unstable mode with purely imaginary frequency for any nonzero \( \mu \).

The approach developed above can be applied not only for solitary waves, but also in the simpler cases, when the equilibrium under consideration is a continuous wave. To illustrate this, one can consider the Hamiltonian describing degenerate intracavity three wave mixing [18]

\[
H = \theta_{11} |\psi_{1}|^2 + \theta_{21} |\psi_{2}|^2 + \frac{1}{2} (\psi_{1}^{\dagger} \psi_{2}^{\dagger} + c.c.) - (\mu_{1} |\psi_{1}|^2 + \mu_{2} |\psi_{2}|^2 + c.c.).
\]  

(21)

Here \( \theta_{11}, \theta_{21} \) are detunings of the fields at frequencies \( \Omega \) and \( 2 \Omega \) from the nearest cavity resonances, \( \mu_{1} = 0 \) and \( \mu_{2} = 0 \) correspond, respectively, to the case of frequency downconversion and second-harmonic generation. For \( \mu_{1,2} = 0 \) the system has gauge symmetry \( (\psi_{1}, \psi_{2}) \rightarrow (\psi_{1} e^{i \theta}, \psi_{2} e^{i \delta \phi}) \). Introducing parameters \( \theta \) and \( \delta \), \( \theta_{11} = \theta \), \( \theta_{21} = 2 \theta + \delta \), one can show that the VK threshold is given by \( \theta_{11}(\psi_{1}|^2 + 2 |\psi_{2}|^2) \) \( = 0 \), and for the solution \( |\psi_{1}|^2 = 2 \theta_{11}^2 |\psi_{2}|^2 \), \( |\psi_{2}| = |\theta_{11}| \), it exists at \( \theta_{21} = 0 \). Therefore VK modes are present for \( \theta_{21} \neq 0 \). If any or both \( \mu_{1,2} \neq 0 \), then complex-frequency instability is expected and indeed happens as was found by traditional analysis [18]. If diffraction is included into Eq. (21), then the same instability of solitary structures can also be found [19].

An important ingredient, which we have disregarded above, is non-Hamiltonian perturbations, the simplest of which are linear losses. A phenomenological account of the effect of losses on the soliton spectrum can be made by replacing \( \omega \) by \( \omega + i \gamma \), \( \gamma > 0 \). This trick is often used, e.g., in textbooks on nonlinear optics, to include damping phenomena into dispersion coefficients. In soliton problems it describes a shift of the soliton spectrum by \( \gamma \) in the \( (Re \omega, Im \omega) \) plane, which qualitatively agrees with numerical calculations [19]. An approach to the problem of non-Hamiltonian perturbations, emphasizing the role played by the energy of internal modes has been outlined in [20] and should be a guideline for future rigorous studies of the interplay between symmetry breaking and losses. Note, that previous works known to the author that explore complex-frequency instabilities due to symmetry breaking, see [19,21,22], do not contain energy based analyses, showing, however, generality of the phenomenon described above.

In summary, considering a class of Hamiltonian systems with broken symmetries, we have demonstrated that internal modes of nonlinear waves, replacing Goldstone modes, carry energy with signs that are independent of the choice of the symmetry-breaking perturbations. In particular, breaking of the phase symmetry in the NLS equation leads to appearance of the negative energy modes, which, in turn, explains the presence of complex frequency instabilities of solitary waves in a variety of physical systems reducible to the driven NLS.

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