Vector modulational instabilities in ultra-small core optical fibres

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Received 13 October 2003, accepted for publication 15 January 2004
Published 10 February 2004
Online at stacks.iop.org/JOptA/6/301 (DOI: 10.1088/1464-4258/6/4/002)

Abstract
We present a detailed analysis of vector modulational instabilities in ultra-small core optical fibres. The existence of new instability peaks emerging due to a strong waveguide contribution to the dispersion characteristic of these fibres is reported and their properties are analysed for the cases of low and high birefringence.

Keywords: modulational instability, four-wave mixing, parametric instabilities

1. Introduction
Modulational instability (MI) of optical waves is a phenomenon determined by the interplay between dispersive and nonlinear properties of a medium in which waves propagate. MI can develop in space and time separately or simultaneously, depending on the experimental set-up. Temporal MI of optical waves has been particularly well studied in telecommunication fibres; see [1] for a detailed review of the subject.

In order to enhance the effective nonlinear coefficient, a conventional telecommunication fibre can be tapered to a thin silica strand with micron-scale diameter [2]. This tapering also leads to a strong modification of the dispersion characteristics of the fibre. Unlike standard fibres with core diameter \( \sim 9 \mu m \) and only one frequency at which the group velocity dispersion (GVD) changes sign, the tapered fibres with core diameter \( \sim 1 \mu m \) have two zero-GVD points [3]. Strong modification of the dispersion characteristic happens due to the reduced diameter of the guiding area, which is now of the order of the optical wavelength, and due to the guiding by the high contrast of the refractive indices of glass and air. The frequency region between the two zero-GVD points exhibits anomalous GVD, and therefore the conditions for the existence of MI are expected to be satisfied. Furthermore, photonic crystal fibres (PCFs) with core diameters of the order of a micron surrounded by large air holes have dispersion characteristics similar to those of tapered fibres; see, e.g. [4, 5]. A major advantage of PCFs over tapered fibres is that their photonic crystal cladding structure protects the core from mechanical damage, which restricts the practical lengths of the tapered fibres to tens of centimetres at best.

It has been previously demonstrated that the presence of the two zero-GVD points significantly alters MI in optical fibres with idealized parabolic frequency dependence of the GVD leading to excitation of secondary MI peaks [6]. For GVD profiles typical of small core PCFs and tapered fibres the existence of similar secondary MI peaks has been reported in [5]. However, in the last case some interesting features emerge due to deviations of the realistic profiles from the parabolic shape [5, 7]. In particular, these deviations lead to the existence of MI of the pump wave with frequency inside the region of normal GVD [5, 7]. The position of the secondary MI peaks in this case is much more sensitive to the value of the pump frequency, compared to the standard situation with the pump belonging to the anomalous GVD region. This property suggests possible applications for frequency conversion devices based on small core fibres; see [8] for some preliminary results in this direction.

Our aim here is to develop a theory of MI in small core fibres that takes into account the vector nature of optical waves. By analogy with conventional fibres [1], we expect that polarization-related instabilities should provide more flexibility in the choice of the pump frequency and in the control of the positions of the generated side-bands. Experimental and theoretical results on MI in PCFs which are relevant to our work have been previously presented in [9]. However, the results of [9] did not go beyond the MI scenario known for conventional fibres. This can apparently be explained by the fact that the second zero-GVD point was either absent or shifted into the far infrared in the PCF used in [9].
2. Governing equations

It has already been shown that the side-bands generated in the case of scalar MI in small core fibres can have detunings of the order of the pump frequency [5, 7]. In this case, a natural question arises concerning the adequacy of the various versions of the nonlinear Schrödinger equation traditionally used to study MI in optics. A comparative study of MI using the scalar wave and the generalized NLS equations [5] has revealed that the NLS model predicts the positions of the strongly detuned MI bands with a very high degree of accuracy and it only fails in the prediction of the relative strengths of the generated bands. In what follows we focus on the prediction of the positions of the MI bands without analysing their relative intensities; therefore we can rely on the vector NLS equations, which include all relevant dispersion orders [1, 5, 7]:

\[ i(\partial_t + \Delta \beta_1 \partial_z) A_x + \frac{1}{2} \Delta \beta \partial_x A_x - \hat{D}_x(i\hat{\beta}_x) A_x + \gamma_1(\|A_x\|^2 + \frac{1}{2}\|A_x\|^2) A_x + \frac{1}{2} A_x^* A_x^2 = 0, \]

\[ i(\partial_t - \Delta \beta_1 \partial_z) A_y - \hat{D}_y(i\hat{\beta}_y) A_y + \gamma_1(\|A_y\|^2 + \frac{1}{2}\|A_y\|^2) A_y + \frac{1}{2} A_y^* A_y^2 = 0. \]

Here, \(A_{x,y}\) are the slowly varying amplitudes of the electric field along the two polarization axes (from now on we use \(\hat{x}\) to denote the slow axis and \(\hat{y}\) to denote the fast axis), \(\gamma_{1,x}\) are the nonlinear coefficients, \(\Delta \beta > 0\) is the propagation constant mismatch between the slow and the fast axes, \(\Delta \beta = (\Delta \beta_{1x} - \Delta \beta_{1y})/2\) is the ‘group velocity’ mismatch, and \(\hat{D}_{x,y}(i\hat{\beta})\) are linear operators describing the dispersions of order higher than first. Note that these operators in the Fourier space are not the Taylor expansions of the dispersions near some reference frequency, but polynomial fits centred at some reference frequency \(\omega_0\): \n
\[ \hat{D}_{x,y}(i\hat{\beta}) = \hat{D}(i\hat{\beta}) = \sum_{m\geq 2} \frac{\beta_m}{m!} (i\hat{\beta})^m. \]

To find the dispersion coefficients \(\beta_{m\geq 2}\), the frequency dependence of the fibre GVD, \(\beta_2(\omega)\), is expressed by the polynomial fit centred at some reference frequency \(\omega_0\):

\[ \beta_2(\omega) = \sum_{m\geq 2} \frac{1}{(m-2)!} \beta_m(\omega_0)(\omega - \omega_0)^{m-2}. \]

Below we always choose the reference frequency to be the pump frequency and recalculate the dispersion coefficients correspondingly. As our representative example we use the tenth-order \((M = 12)\) polynomial fit to the numerically computed GVD profile of a 1 \(\mu\)m tapered fibre; see figure 1. We now give the dispersion coefficients calculated for \(\omega_0 = 350\) THz, which are \(\beta_2 = -0.115\) ps\(^2\) m\(^{-1}\), \(\beta_3 = 6.09 \times 10^{-4}\) ps\(^3\) m\(^{-1}\), \(\beta_4 = 1.90 \times 10^{-6}\) ps\(^4\) m\(^{-1}\), \(\beta_5 = -2.97 \times 10^{-8}\) ps\(^5\) m\(^{-1}\), \(\beta_6 = 3.495 \times 10^{-10}\) ps\(^6\) m\(^{-1}\), \(\beta_7 = -3.03 \times 10^{-12}\) ps\(^7\) m\(^{-1}\), \(\beta_8 = 1.518 \times 10^{-14}\) ps\(^8\) m\(^{-1}\), \(\beta_9 = -4.336 \times 10^{-17}\) ps\(^9\) m\(^{-1}\), \(\beta_{10} = 7.025 \times 10^{-20}\) ps\(^{10}\) m\(^{-1}\), \(\beta_{11} = -6.026 \times 10^{-23}\) ps\(^{11}\) m\(^{-1}\), \(\beta_{12} = 2.127 \times 10^{-26}\) ps\(^{12}\) m\(^{-1}\).

Values of other parameters we use in our estimates are \(\gamma_{1,x} = \gamma = 0.165\) m\(^{-1}\) W\(^{-1}\), \(\Delta \beta = 6.2\) m\(^{-1}\) (which corresponds to a beat length of \(\sim 1\) m) and the group velocity mismatch parameter \(\Delta \beta_1\) is assumed to be 0.1 ps m\(^{-1}\). In the last section we also consider larger values of \(\Delta \beta\) in order to study the transition from the low birefringence to the high birefringence scenario.

3. Linear stability analysis

We seek CW solutions of equations (1) in the form

\[ A_x = \sqrt{P_x} \exp(i\phi_0 x + \frac{i}{2} \Delta \beta \phi_0^2), \]

\[ A_y = \sqrt{P_y} \exp(i\phi_0 x + i\alpha), \]

where \(\alpha\) and \(\phi_0,0)\) are real phases and \(P_{x,y}\) are the powers of the optical fields. After substitution of equations (4) into (1), and assuming \(\gamma_4 = \gamma_5 = \gamma\), we find

\[ \sqrt{P_x}\{\gamma(P_x + \frac{1}{2} P_y) + \frac{1}{2} \gamma P_y \times \exp[-2i(z(\phi_0 - \phi_0) + 2i\alpha) + \frac{1}{2} \Delta \beta]\} = \sqrt{P_x} \phi_0, \]

\[ \sqrt{P_y}\{\gamma(P_y + \frac{1}{2} P_x) + \frac{1}{2} \gamma P_x \times \exp[+2i(z(\phi_0 - \phi_0) - 2i\alpha) - \frac{1}{2} \Delta \beta]\} = \sqrt{P_y} \phi_0. \]

\(\phi_0 = \phi_0 \equiv \phi\) is the condition of \(z\)-independence of equations (4), which needs to be imposed due to the phase sensitive nature of the coherent interaction. Also the right-hand sides of equations (5) are real, and so have to be the left-hand sides; this condition can be satisfied for \(\alpha = 0, \pm \pi/2\). These values of \(\alpha\) are the only ones resulting in physically distinct polarization states, which we consider in detail below.

For \(\alpha = 0\) equation (5) has two obvious linearly polarized eigensolutions with phases and powers given by

\[ \gamma P_{x,y} + \frac{1}{2} \Delta \beta = \phi, \quad P_{x,y} = 0. \]
Inserting $\alpha = \pm \pi/2$ into equation (5) we have the following condition for the existence of elliptically polarized stationary solutions:

$$P_\alpha = P_Y - P_s,$$

(7)

where $P_\alpha$ is a ‘critical power’, usually associated with the polarization instability phenomenon [1], given by $P_\alpha = 3\Delta \beta /2\gamma$. From equation (7) it follows that in the elliptically polarized steady-state solution the fast axis always carries more power than the slow axis. While CW solutions (6), (7) are exactly the ones known for conventional fibres [1], their stability properties will be strongly affected by the distinct dispersion characteristics of the small core fibres, and this makes the difference between our work and previous extensive works on the subject [10–14].

To find the spectra of instabilities of the stationary solutions of equations (1) we perturb each component of the steady-state solutions with small amplitude complex fields $u$ and $v$:

$$A_x \rightarrow \left[ \sqrt{P_x + \epsilon u(z, t)} \right] \exp(i\phi_{0x} z),$$

$$A_y \rightarrow \left[ \sqrt{P_y + \epsilon v(z, t)} \right] \exp(i\phi_{0y} z + i\alpha).$$

(8)

Substituting equations (8) into the original equations, linearizing with respect to the small parameter $\epsilon$, and using the following expressions for the perturbation fields:

$$u(z, t) = u_{AS}(z) \exp(-i\delta t) + u_S(z) \exp(i\delta t),$$

$$v(z, t) = v_{AS}(z) \exp(-i\delta t) + v_S(z) \exp(i\delta t),$$

(9)

we obtain the equation governing the evolution of the Stokes and anti-Stokes components of the vector perturbation field

$$\partial_x \hat{\psi} = i\hat{M} \hat{\psi}, \quad \hat{\psi} \equiv (u_S, u_{AS}, v_S, v_{AS})^T.$$  

(10)

Assuming that $\hat{\psi} \sim e^{i\phi_{0} z}$ one readily shows that a CW is stable providing that characteristic equation det$(\hat{M} - \kappa I) = 0$ has no roots with negative imaginary parts. Those frequency regions where the imaginary part of $\kappa_j(\delta)$ is negative clearly correspond to the instability bands. Subsequent analysis is divided into three parts: (a) stability for the $x$-polarized slow wave, (b) stability of the $y$-polarized fast wave, and (c) stability of the elliptically polarized state described by (7). The explicit forms of the real $4 \times 4$ matrix $\hat{M}$ for all the relevant cases will be given below.

### 3.1. Instabilities of the slow pump wave

In this case the stability matrix has the following diagonal structure:

$$\hat{M} = \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{B} \end{pmatrix},$$

(11)

where

$$\hat{A} = \begin{pmatrix} \gamma P_s + \Delta \beta_1 \delta + D_{-}(-) & \gamma P_s \\ -\gamma P_s - \Delta \beta_1 \delta - D_{+} & -\gamma P_s \end{pmatrix},$$

and

$$\hat{B} = \begin{pmatrix} \frac{1}{2} \gamma P_s + \Delta \beta_1 \delta + D_{-} & \frac{1}{2} \gamma P_s \\ -\frac{1}{2} \gamma P_s - \Delta \beta_1 \delta - D_{+} & -\frac{1}{2} \gamma P_s \end{pmatrix}.$$  

(12)

Here we introduced the following notation:

$$D_\pm = \sum_{m \geq 2} \frac{(\pm 1)^m \beta_m}{m!} \delta^m.$$  

(13)

The four eigenvalues $\kappa_{1,2,3,4}$ of matrix $\hat{M}$ are given by

$$\kappa_{1,2} = -\Delta \beta_1 \delta - \frac{1}{2}[D_+ - D_-]$$

$$\pm \sqrt{\frac{D_+ + D_-}{2} - \Delta \beta} \left( \frac{D_+ + D_-}{2} + 2\gamma P_s \right).$$

(14)

$$\kappa_{3,4} = \Delta \beta_1 \delta - \frac{1}{2}[D_+ - D_-]$$

$$\pm \sqrt{\left( \frac{D_+ + D_-}{2} - \Delta \beta \right) \left( \frac{D_+ + D_-}{2} + \frac{2}{3} \gamma P_s - \Delta \beta \right)}.$$  

(15)

(16)

Matrix $\hat{A}$ is the stability matrix that has been previously derived in the scalar analysis [5, 7]. Matrix $\hat{B}$ represents the contribution to the instability due to the presence of the orthogonally polarized fast wave.

Figure 2(a) shows the imaginary parts of all of the non-vanishing eigenvalues $\kappa_j$ of the matrix $\hat{M}$ when fibre is pumped in the region of anomalous GVD, while figure 2(b) shows the case where the fibre is pumped in a region of normal GVD. When the MI-generated waves are linearly polarized we denote them in all the figures with the letters ‘S’ and ‘P’. ‘S’ stands for ‘scalar instabilities’, i.e. for the instabilities which have already been described in [5, 7], and ‘P’ stands for polarization instabilities. Analogues of MI bands nearest to $\delta = 0$ (see figure 2) can be found in standard telecommunications fibres [1], while all the far detuned bands are specific to the small core fibres. These new peaks are attributable to the four-wave mixing process, which is phase matched by the linear refractive index, and it is due to the existence of the low frequency zero-GVD point with negative slope of $\beta_2$ [5].

The behaviour of the MI bands when fibre is pumped in the vicinity of the two zero-GVD points, see figure 3 (solid lines), is particularly interesting because it clearly demonstrates the role played by the dispersions of orders higher than fourth [5]. In particular, working in the proximity of the high frequency zero-GVD point, we find that detunings of the slow and fast MI peaks change rapidly with relatively small shifts of the pump frequency; see figure 3(b). In contrast, in the proximity of the low frequency zero-GVD point this dependence is smooth (see figure 3(a)), and therefore less interesting for potential frequency conversion applications.

### 3.2. Instabilities of the fast pump wave

In the case (b) the pump wave is polarized along the fast $y$-axis and the matrices $\hat{A}$ and $\hat{B}$ in equation (11) have the following forms:

$$\hat{A} = \begin{pmatrix} \frac{1}{2} \gamma P_y + \Delta \beta - \Delta \beta_1 \delta + D_{-} & \frac{1}{2} \gamma P_y \\ -\frac{1}{2} \gamma P_y - \Delta \beta + \Delta \beta_1 \delta - D_{+} & -\frac{1}{2} \gamma P_y \end{pmatrix},$$

(17)

and

$$\hat{B} = \begin{pmatrix} \frac{1}{2} \gamma P_y + \Delta \beta_1 \delta + D_{-} & \frac{1}{2} \gamma P_y \\ -\frac{1}{2} \gamma P_y - \Delta \beta_1 \delta - D_{+} & -\frac{1}{2} \gamma P_y \end{pmatrix}.$$  

(18)
Figure 2. MI growth rates for the pump wave polarized along the slow axis. Letters ‘S’ and ‘P’ mark scalar and polarization instabilities, respectively. (a) The pump frequency $\omega_0 = 260$ THz belongs to the region of anomalous GVD. (b) The pump frequency $\omega_0 = 555$ THz belongs to the region of normal GVD. Other parameters are: pump power $P_p = 200$ W, $\Delta \beta_1 = 6.2 \text{ m}^{-1}$, $\Delta \beta_1 = 0.1 \text{ ps m}^{-1}$, $\gamma = 0.165 \text{ m}^{-1} \text{ W}^{-1}$. The part of the spectrum with negative $\delta$ is omitted because it is just a mirror image of the $\delta > 0$ part.

$\hat{B}$ now describes the scalar instabilities and $\hat{A}$ is responsible for the excitation of the orthogonally polarized slow $x$-waves. The four eigenvalues $\kappa_{1,2,3,4}$ of $\hat{M}$ are given by

$$\kappa_{1,2} = -\Delta \beta_1 \delta - \frac{1}{2} [D_s - D_-] + \sqrt{\left( D_s + D_- + \Delta \beta \right) \left( D_s + D_- + \frac{2}{3} \gamma P_s + \Delta \beta \right)}.$$ (19)

$$\kappa_{3,4} = \Delta \beta_1 \delta - \frac{1}{2} [D_s - D_-] + \sqrt{\left( D_s + D_- + \Delta \beta \right) \left( D_s + D_- + 2 \gamma P_s \right)}.$$ (20)

Typical instability spectra for pump frequencies belonging to regions of anomalous and normal GVDs are shown in figures 4(a) and (b), respectively. The behaviour of the MI peaks in the vicinity of the zero GVD points is shown in figure 3 (dotted curves). Here again all far detuned narrow peaks are absent in conventional fibres. The behaviour of MI bands in the proximity of the zero-GVD points is similar to the case for the slow pump.

3.3. Instabilities of the elliptically polarized pump wave

Now we consider the case of the elliptically polarized pump beam, i.e. when the phase difference $\alpha$ in equation (4) is $\pm \pi/2$. The stability matrix $\hat{M}$ contains anti-diagonal submatrices $\hat{C}$ and $\hat{D}$:

$$\hat{M} = \begin{pmatrix} \hat{A} & \hat{C} \\ \hat{D} & \hat{B} \end{pmatrix},$$ (21)

where

$$\hat{A} = \begin{pmatrix} \gamma P_s + \frac{1}{2} \gamma P_s - \Delta \beta_1 \delta + D_- & -\gamma P_s \\ -\gamma P_s & \gamma P_s - \Delta \beta_1 \delta - D_- \end{pmatrix},$$ (22)

$$\hat{B} = \begin{pmatrix} \frac{1}{2} \gamma P_s + \gamma P_s + \Delta \beta_1 \delta + D_- & -\gamma P_s \\ -\gamma P_s & \frac{1}{2} \gamma P_s - \gamma P_s + \Delta \beta_1 \delta - D_- \end{pmatrix},$$ (23)

$$\hat{C} = \hat{D} = \begin{pmatrix} \frac{1}{2} \gamma \sqrt{P_s P_s} & \frac{1}{2} \gamma \sqrt{P_s P_s} \\ -\frac{1}{2} \gamma \sqrt{P_s P_s} & -\frac{1}{2} \gamma \sqrt{P_s P_s} \end{pmatrix}.$$ (24)
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To launch the elliptically polarized eigensolution into the fibre one should balance the ratio of $P_x$ and $P_y$ determined by the birefringence parameter $\Delta n$ via equation (7). This creates additional difficulties for experimental observation of the instabilities described below, though it does not make them less significant from a fundamental point of view.

In figure 5(a) we show the MI gain that we have obtained in the case of anomalous dispersion. In this case, surprisingly, the system is stable around $\delta = 0$, even though the GVD at the pump frequency is anomalous. The polarization states of the excited waves are in general elliptical. When the GVD at the pump frequency is normal we have a similar instability plot to that in the case of anomalous GVD, but with the first set of peaks missing, and with a more complicated secondary-peak structure; see figure 5(b). The behaviour of the MI peaks of the elliptically polarized eigensolution in the vicinity of the zero-GVD points is shown in figure 3 with dashed curves.

For the linearly polarized pump the coherent coupling between polarizations gradually loses its importance when the birefringence parameter is increased and therefore all polarization instabilities gradually vanish in full analogy with conventional fibres [1]. In the case of elliptically polarized solutions, however, the coupling between the orthogonally polarized modes is enforced by the two-component nature of the pump wave itself. Therefore multiple peaks that are near and far detuned are preserved even for the strong birefringence case. An example of the transformation of the far detuned peaks for increasing birefringence parameter is shown in figure 6.

4. Summary

In summary, we have analysed MI of different polarization eigenstates in ultra-small core silica fibres and demonstrated the existence of new—compared to the case for conventional telecommunications fibres—bands of polarization instabilities. The existence of these bands extends the opportunities for applications of small core fibres for frequency conversion purposes [8], which are partially known from the scalar analysis [5, 7].

Acknowledgments

We acknowledge several useful discussions with J M Pottage, W J Wadsworth and A V Yulin.
Figure 6. The splitting of the far detuned MI peaks from figure 5(a) into a triplet during the transition from the low to the high bi-refringence scenario.

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