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The Classification of Isotropic Points in Stress Fields*

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ABSTRACT

Isotropic points are significant features of any complicated two-dimensional field of stress and strain, for they are stable against perturbation. Around them the trajectories of principal stress or strain have three patterns, rather than the two usually recognized, the extra pattern being called monstar. The example of three-point bending of a beam illustrates how isotropic points can be born in pairs, one member of the pair necessarily having the monstar pattern, and how a point can subsequently change from one pattern to another. By making yet another distinction, one can divide isotropic points into six categories, in general. In the special case of statical equilibrium without body forces, the number of different categories for the isotropic points in a stress distribution reduces to four and there is a close analogy with the umbilic points of a surface, but for strain the number remains six.

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I. INTRODUCTION

In two-dimensional distributions of stress and strain there are, in general, special points where the two principal stresses or strains are equal. These are the isotropic points of photoelasticity, for which numerous examples are readily visible in any irregularly shaped photoelastic specimen under load. In fact, the complicated stress distribution that is produced in such a specimen can usefully be thought of as being based on the positions and nature of these points. This is because isotropic points have the important property of being *structurally stable*; that is to say, a small change in the loading or in the shape of the specimen does not destroy them, but merely moves them slightly. Thus a complicated change in the pattern may be categorized, in one way, by simply noting the movement of the isotropic points. They form a kind of skeleton on which the rest of the stress field is built. It is therefore of both practical and theoretical importance to understand their nature. The purpose of this paper is to bring to the attention of engineers certain recent theoretical results on this question.

Traditionally, isotropic points have been classified simply as positive or negative, each kind having its characteristic pattern of stress trajectories. We point out in Section II that this is an error, because there are actually three characteristic patterns rather than two. In practice it is hard to see the details near an isotropic point, because the principal stress difference is small. This is why erroneous patterns are often drawn. To avoid mistakes it is helpful to know, from theory, what to expect.

These preliminary remarks are intended to demonstrate the relevance to engineering of the formal classification which follows. A further small illustration of the practical significance of isotropic points is the following: Suppose a plate is to be subjected to plane stress or plane strain and one wishes to bore small bolt holes in it that will remain circular after deformation. A small circular hole made at a general point will become elliptical. To solve the problem, one should place the holes at the isotropic points, for they will then remain circular (if small enough) under the prescribed load, even though they will be expanded or contracted.

After describing the classification of isotropic points in Section II, we give in Section III a simple example, three-point bending, where they play a prominent role. In particular, the example shows how isotropic points can be born in pairs. Whenever this happens, the stress pattern around one member of the pair has to be of the unfamiliar monstar variety.

II. CLASSIFICATION OF ISOTROPIC POINTS

The following classification scheme applies to the isotropic points in any two-dimensional field of a symmetric tensor, but for definiteness we shall first describe it for the stress tensor. It was introduced by Thorndike, Cooley, and Nye [1] to describe the "generalized umbilic points" that occur in a general two-dimensional vector field, the analysis for this case being essentially identical with that for a symmetric tensor. This work was itself based on the related classification given by Berry and Hannay [2] for the umbilic points of a general surface (points where the two principal curvatures are equal). Further details of the scheme will be found in these two papers.

The classification may be explained by referring to Fig. 1, which shows three possible patterns for the trajectories of principal stress around an isotropic point. At each point of a pattern, the light solid line shows the direction of the algebraically greater of the two principal stresses and the orthogonal light dashed line shows the direction of the other principal stress. In each pattern the principal stress directions become indeterminate at the isotropic point itself. These three patterns are special because they are symmetrical. A general isotropic point will have a stress pattern that is topologically similar to one of these, but is distorted.

The classification of isotropic points in general depends on recognizing that any given isotropic point possesses three distinct properties. The first property is the *disclination index*, or simply the *index*. This results in the traditional division into positive and negative isotropic points. To find the index, make an imaginary clockwise circuit around the point and note the corresponding rotation (clockwise positive) of the principal stress directions. If it is $+\pi$

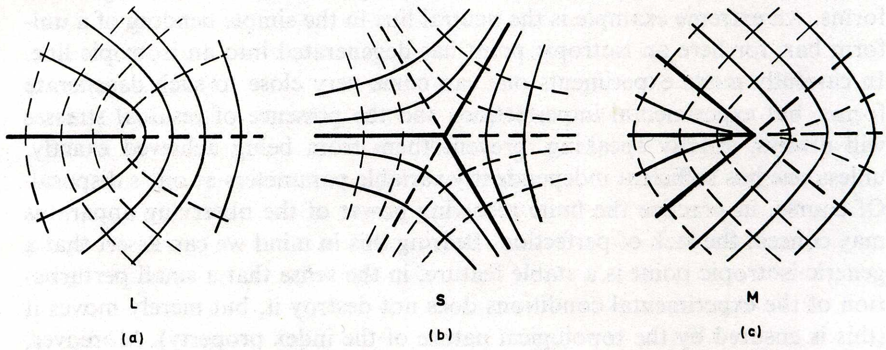


Fig. 1 The pattern or principal stress directions for (L) lemon, (S) star, and (M) monstar. Note that in the monstar pattern all the lines from a whole sector pass through the isotropic point, while in the lemon pattern only one line does so.

the index is $+\frac{1}{2}$, if it is $-\pi$ the index is $-\frac{1}{2}$. Thus, in Fig. 1b the index is $-\frac{1}{2}$, while in Figs. 1a and 1c it is $+\frac{1}{2}$.

The second property is the *line* property. This is one or three, depending on the number of straight stress trajectories passing through the point ("straight" means that the curvature vanishes in the lowest approximation). In Fig. 1 the straight trajectories are shown by thick solid and dashed lines. Thus in Fig. 1a the line property is one, while in Figs. 1b and 1c it is three. These examples show that classifying by the index property is different from classifying by the line property. However, out of the four combinations of these two properties, one is impossible: negative index and one line cannot occur simultaneously. Figures 1a, 1b, and 1c are in fact symmetrical examples of the three remaining combinations. The pattern of Fig. 1a (index $+\frac{1}{2}$, 1 line) is called lemon (L), Fig. 1b (index $-\frac{1}{2}$, 3 lines) is called star (S), and Fig. 1c (index $+\frac{1}{2}$, 3 lines) is called monstar (M) because it is intermediate between lemon and star. The nomenclature used here is due to Berry and Hannay [2].

The third property is different from the others, in that it does not involve the stress trajectories. Instead, it depends on the shape of the contours of constant principal stress σ_1 and σ_2; i.e., on the shape of the curves $\sigma_1 =$ constant, $\sigma_2 =$ constant near the isotropic point. The curves are either both ellipses or both hyperbolas. The *contour* property is elliptic (E) if the contours are ellipses and hyperbolic (H) if they are hyperbolas. Each of the line patterns of Fig. 1 can be elliptic or hyperbolic, in this sense. There are thus six different possibilities in all.

Isotropic points in stress distributions are *generic*, that is, they will occur naturally without any special preparation (for example, a symmetrical shape or loading) being necessary. In this paper, we are primarily concerned with what happens in such "general" fields. In theory, one can construct degenerate forms. An extreme example is the neutral line in the simple bending of a uniform bar, for here an isotropic point has degenerated into an isotropic line. In carefully made experiments one can come very close to such degenerate forms, but experimental imperfections and the presence of residual stresses will always, strictly speaking, prevent them from being achieved exactly, unless one has sufficient independently variable parameters at one's disposal. Of course, in practice the finite resolving power of the observing apparatus may conceal the lack of perfection. Bearing this in mind we can assert that a generic isotropic point is a stable feature, in the sense that a small perturbation of the experimental conditions does not destroy it, but merely moves it (this is ensured by the topological nature of the index property). Moreover, its classification is a stable property; when slightly perturbed the point retains both its identity and its classification. On the other hand, a large perturbation may shift it from one category to another.

Certain degenerate special cases can be achieved in practice (as well as in theory) as transitional forms that appear as a single parameter (for example, the aspect ratio of a specimen) that is continuously changed. The borderline forms between monstar and lemon and between elliptic and hyperbolic can be made in this way, but they are nongeneric (they only occur for a special value of the parameter) and unstable; a small perturbation will change them into one of the stable categories. In the same way, the symmetry of the patterns in Fig. 1 is nongeneric; typical isotropic points have unsymmetrical trajectory patterns.

All three properties of an isotropic point (index, line, and contour) depend on the linear terms in the expansion of the stress tensor about the isotropic point, which may be written up to linear terms as

$$\sigma = \begin{pmatrix} \sigma_0 + ax + by & -d_1x - a_1y \\ -d_1x - a_1y & \sigma_0 + cx + dy \end{pmatrix}$$

where x and y are Cartesian coordinates and σ_0, a, a_1, b, c, d, and d_1 are constants. The sign of the index is the same as the sign of the discriminant (Ref. 1, p. 1489)

$$D_I = (c - a)a_1 + (b - d)d_1 \tag{1}$$

The line property depends on the sign of the discriminant [1]

$$D_L = 4\{3a_1(c - a - a_1) - (b - d - d_1)^2\}\{3d_1(b - d - d_1) - (c - a - a_1)^2\} - \{(b - d - d_1)(c - a - a_1) - 9d_1a_1\}^2 \tag{2}$$

If $D_L > 0$, there are three lines, and if $D_L < 0$, there is one line. The contour property depends on the sign of the discriminant [1]

$$D_C = 4(ca - d_1^2)(db - a_1^2) - (cb + da - 2d_1a_1)^2 \tag{3}$$

The point is elliptic if $D_C > 0$ and hyperbolic if $D_C < 0$.

If the body is in statical equilibrium and there are no body forces, the equilibrium equations for stress demand that $a = a_1$ and $d = d_1$. In this special case, the stress components are derivable from the Airy stress function $\chi(x, y)$ by the relations $\sigma_{xx} = \partial^2\chi/\partial y^2$, $\sigma_{yy} = \partial^2\chi/\partial x^2$, and $\sigma_{xy} = -\partial^2\chi/\partial x\partial y$. If $\chi(x, y)$ is regarded as the height of a surface above the (x, y) plane (scaling χ so that the surface has only small slopes), the curvature tensor of the surface has the form

$$\begin{pmatrix} \partial^2\chi/\partial x^2 & \partial^2\chi/\partial x\partial y \\ \partial^2\chi/\partial x\partial y & \partial^2\chi/\partial y^2 \end{pmatrix} = \begin{pmatrix} \sigma_0 + cx + dy & dx + ay \\ dx + ay & \sigma_0 + ax + by \end{pmatrix}$$

At (0, 0) the two principal curvatures are equal, namely σ_0, and the point is, by definition, umbilic. Berry and Hannay [2] classify the umbilic points of a general surface by index, line, and contour, just as we have done above for isotropic points, the curvature tensor playing the part of the stress tensor. It may be verified (using Ref. 2) that the discriminants are then precisely given by Eqs. 1, 2, and 3, but with $a = a_1$ and $d = d_1$. Thus, where the tensor components are derivable from an underlying potential (here the Airy stress function), the classification of a given isotropic point of the tensor field is identical with the classification for the corresponding umbilic point of the potential surface. Where there is no such potential, the more general classification (with $a \neq a_1$ and $d \neq d_1$) relevant to "generalized umbilic points" is needed.

When a potential exists, two of the six classes of isotropic point are forbidden, namely elliptic lemon and elliptic monstar, and the four remaining classes may be displayed on the one-dimensional Venn diagram in Fig. 2. Thus, for example, all elliptic patterns have negative index and three lines, while all one-line patterns are hyperbolic and have positive index.

To distinguish experimentally (say, by photoelasticity) between positive and negative isotropic points, i.e., to determine the index property, is comparatively simple. It can be much harder to distinguish between lemon and monstar and this is doubtless the reason why monstar patterns are so rarely reported (an example occurs in Fig. 7 of Ref. 3). We suspect that many patterns drawn as lemons are in fact monstars. However, it is worth remarking that on an isotropic Gaussian random surface monstar umbilic points are

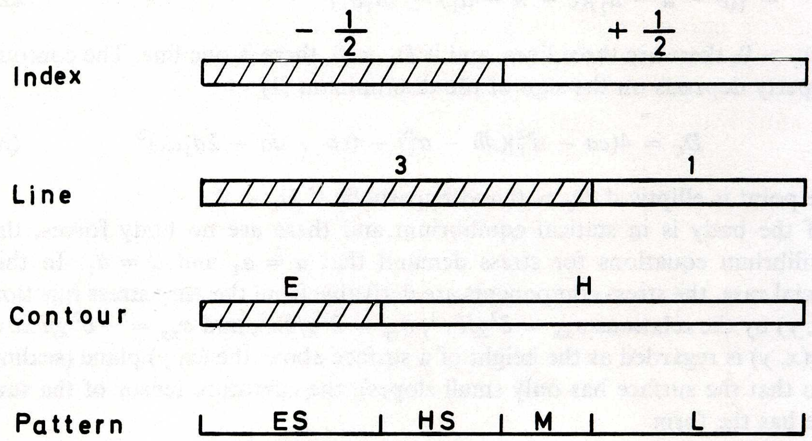


Fig. 2 Classification for the isotropic points of a stress distribution when the stresses are in equilibrium. The diagrams show how the various classes overlap. ES = elliptic star, HS = hyperbolic star, M = monstar, L = lemon.

comparatively rare. Berry and Hannay [2] calculate the proportions as 26.8% elliptic star, 23.2% hyperbolic star, 44.7% lemon, and 5.3% monstar. It follows that, if such a random surface is taken to represent an Airy stress function, the resulting stress distribution will contain isotropic points in these same proportions.

The first mention of the monstar pattern seems to be by Darboux [4], who noted it as a possible configuration of the lines of principal curvature of a surface. Filon [5] rediscovered it in photoelasticity, but nevertheless textbooks of photoelasticity still usually illustrate only the star and the lemon.

The index property of an isotropic point means that it is topologically impossible for a single isotropic point to be created in the interior of a stress field where none existed before. As an external parameter is changed, the points are born in pairs (one positive and the other negative), unless they come in from boundaries. If there is an underlying potential (equilibrium), as Fig. 2 indicates, one member of the pair will be hyperbolic star, while the other will be monstar. Similarly, the points can mutually annihilate one another in pairs. Figure 3 shows the corresponding patterns of principal stress directions (these were originally calculated [6] as patterns of lines of principal curvature for the mutual annihilation of two hyperbolic umbilic points on a surface). Reading Fig. 3 from right to left, in the first pattern there is no isotropic point. In the second pattern two isotropic points coincide, the index is zero and the pattern is degenerate. In the third pattern the two isotropic points have separated, the upper one being clearly a star while the lower is a monstar. As the parameter continues to change, the monstar can transform into a lemon. In fact this is the only way of creating a lemon, unless it moves in from a boundary. In theory, of course, one could create a lemon from nothing by gradually switching on a suitable stress field that was initially everywhere zero. But a zero stress field is nongeneric; in practice there will always be a field of residual stresses, however small.

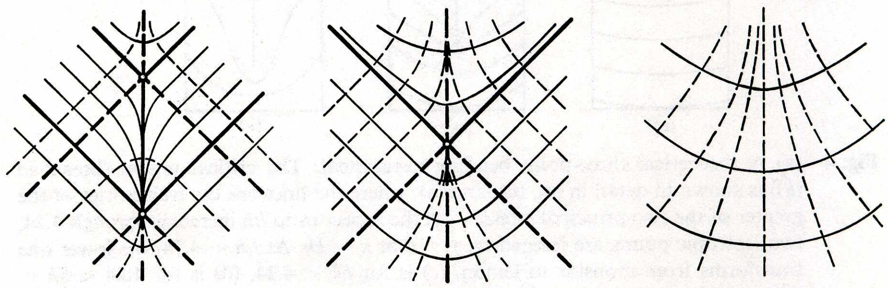


Fig. 3 Sketch of patterns of principal stress directions for successive stages in the mutual annihilation or birth of a pair of isotropic points (symmetrical case).

III. AN ILLUSTRATIVE STRESS FIELD

Exactly the behavior just described occurs in the three-point bending experiment of Fig. 4(a) as the ratio of the length to the depth of the beam is changed. A good approximation to the stresses is given by the Wilson-Stokes theory [7], which predicts that as l/h, where $2l$ is the span and $2h$ is the depth, increases through the value 4.24, two isotropic points are born on the central vertical axis at $x = \frac{1}{2}h$, which then separate in the vertical direction (Figs. 4c, 4d, and 4e). By using the discriminants of Eqs. 1, 2, and 3, with the expressions for the stress components given by Frocht [7], it may be shown that the upper of the two isotropic points is a hyperbolic star, while the lower is a monstar. At $l/h = 4.74$ and $x = 0.68h$, the monstar transforms into a lemon (Fig. 4e).

The essential reason for the appearance of two isotropic points is that, in the top half of the specimen, there is a competition between the longitudinal compression induced by bending and the longitudinal tension induced immediately under the central indenter. The first varies linearly with x but the second does not, thus giving the possibility of cancellation at two different points.

We have made photoelastic observations of a closely related experimental

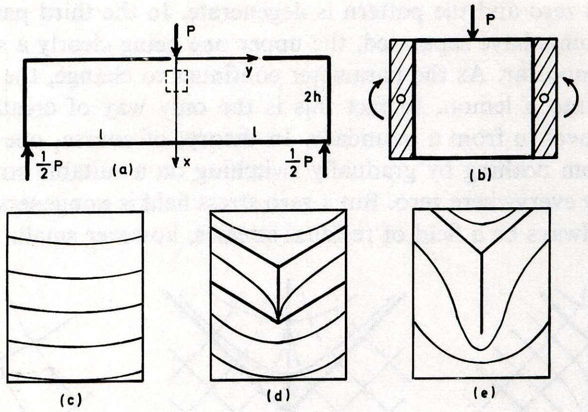
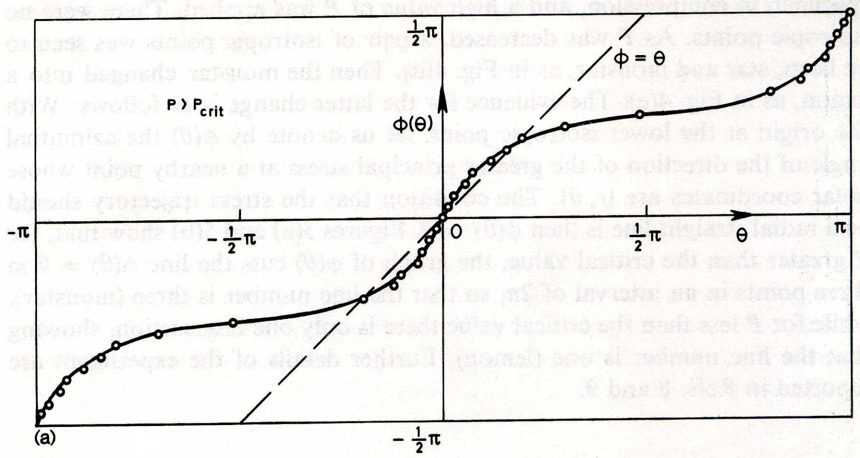


Fig. 4 (a) A theoretical three-point bending experiment. The critical region (boxed in (a)) is shown in detail in (c), (d), and (e), where the lines are the trajectories of the greater of the two principal stresses. As the aspect ratio l/h increases through 4.24, two isotropic points are formed on $y = 0$ at $x = \frac{1}{2}h$. At $l/h = 4.74$, the lower one transforms from monstar to lemon. (c) is for $l/h < 4.24$, (d) is for $4.24 < l/h < 4.74$, and (e) is for $l/h > 4.74$. The experimental arrangement, shown in (b), was slightly different. It produced the same sequence of stress patterns, but by varying the indenter load P rather than by changing the aspect ratio l/h.

arrangement shown in Fig. 4(b). It is like Fig. 4(a) except that the ends of the beam, made of urethane rubber, are cemented to metal bars that are free to rotate about fixed pivots. Thus, applying load P with the metal bars held fixed induces end couples (in contrast to what happens in Fig. 4a). Additional equal and opposite end couples can then be applied by turning the metal bars. In this way, both P and the end couples can be varied independently,



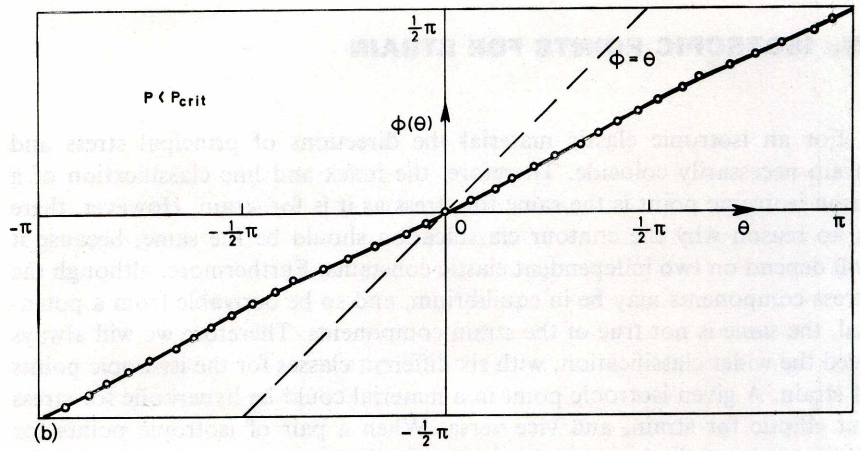


Fig. 5 Results from the experiment shown in Fig. 4(b). The azimuthal angle ϕ of the direction of the greater principal stress is plotted against the polar angle θ for (a) a monstar pattern and (b) a lemon pattern, obtained by decreasing P through a critical value, P_{crit}. The points were obtained by using crossed polarizer and analyzer and measuring the polar angle θ corresponding to the isoclinic labeled ϕ. Thus each measurement is plotted twice, once at θ and once at $\theta + \pi$ according to the relation $\phi(\theta + \pi) = \phi(\theta) + \frac{1}{2}\pi$.

but l/h is fixed (≈ 1). The stress distribution in the center of the beam produced by this arrangement is very similar to that of Fig. 4(a), for both combine the effect of a uniform bending moment with that of an indenter. With the end bars held fixed, decreasing P in Fig. 4(b) corresponds to increasing l/h in Fig. 4(a). This is in fact the experiment that we now report. The end bars were fixed, to give simple bending with the top half of the specimen in compression, and a high value of P was applied. There were no isotropic points. As P was decreased, a pair of isotropic points was seen to be born, star and monstar, as in Fig. 4(d). Then the monstar changed into a lemon, as in Fig. 4(e). The evidence for the latter change is as follows: With the origin at the lower isotropic point, let us denote by $\phi(\theta)$ the azimuthal angle of the direction of the greater principal stress at a nearby point whose polar coordinates are (r, θ). The condition that the stress trajectory should be a radial straight line is then $\phi(\theta) = \theta$. Figures 5(a) and 5(b) show that, for P greater than the critical value, the graph of $\phi(\theta)$ cuts the line $\phi(\theta) = \theta$ in three points in an interval of 2π, so that the line number is three (monstar), while for P less than the critical value there is only one intersection, showing that the line number is one (lemon). Further details of the experiment are reported in Refs. 8 and 9.

IV. ISOTROPIC POINTS FOR STRAIN

For an isotropic elastic material the directions of principal stress and strain necessarily coincide. Therefore, the index and line classification of a given isotropic point is the same for stress as it is for strain. However, there is no reason why the contour classification should be the same, because it will depend on two independent elastic constants. Furthermore, although the stress components may be in equilibrium, and so be derivable from a potential, the same is not true of the strain components. Therefore we will always need the wider classification, with six different classes for the isotropic points of strain. A given isotropic point in a material could be hyperbolic for stress but elliptic for strain, and vice versa. When a pair of isotropic points for strain are created, they must have opposite index; one must be star and the other monstar as in Fig. 3. Either both are elliptic or both are hyperbolic, in contrast to the case for stress in equilibrium with no body forces, when both points must be hyperbolic.

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